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The determination of optimal United States dairy product marketings by quadratic programming

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THE DETERMINATION OF OPTIMAL UNITED STATES DAIRY PRODUCT MARKETINGS BY QUADRATIC PROGRAMMING

by

Gail Eric Updegraff

A Thesis Submitted to the

Graduate Faculty in Partial Fulfillment of

The Requirements for the Degree of

MASTER OF SCIENCE

Major Subject: Agricultural Economics

Signatures have been redacted for privacy

Iowa State University
Of Science and Technology
Ames, Iowa

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I. INTRODUCTION

A. Purpose of Study

The study presented here was undertaken in order to determine price and quantity variables for United States milk and milk products that would have given optimal cash receipts to United States dairy farmers for the years 1951, 1955, 1960, and 1964. The results of such a study can be used to gain policy insights as to how the cash receipts from milk sold by dairy farmers could be increased, thereby increasing dairy farmers' gross incomes. If the present milk marketing situation is not an optimal one with respect to cash receipts of dairy farmers, then the welfare implications and political acceptability of the optimal situation must be considered.

The function of price is to allocate resources and to distribute income among the owners of the various factors of production. It is readily apparent from what has been said in the preceding paragraph that in this study prices will be considered for use as equity instruments rather than allocation instruments. Justification of the use of prices as instruments of equity can be shown by an analysis of available statistics on average income and investment for commercial farms, by a review of a Federal Milk Market Order Program study supported by Associated Dairymen, Inc. (10), and by observation of current dairy farmer dissatisfaction such as is evident from the recent actions of the National Farmers Organization. (The study supported by Associated Dairymen, Inc. was conducted by the Dairy Marketing Advisory Committee, a committee of Land Grant University marketing economists established at the request of Associated Dairymen. The Committee was asked to study and evaluate the Federal Milk Marketing Order Program and offer recommendations for improvement of the system. The study encompassed 66 of a total of 77 Federal Milk Marketing Orders as of 1965. Producer receipts of these 66 Orders amounted to over 30 billion pounds of milk per year, which is about 60 percent of the total milk supply in the Federal Order system.)

The 1964 United States Department of Agriculture statistics (41) on average income and investment of all commercial farms in the United States support the timeliness of this study. For example, in the Central Northeast United States dairy region, one of the best milk producing regions in the country, average net income of dairy farmers dropped from $4,567 in 1958 to $4,178 in 1964. These same farmers had an average investment of $45,500 and an average herd size of 33 in 1964. The average American family had an income of $6,556 in 1964. Grade A milk producers in eastern Wisconsin, the top income receivers among United States dairy farmers, received an average of $6,541 in return for an average investment of $71,950 in 1964. This amount of money invested at four percent interest and compounded semiannually would return $2,906.78. If a farmer chose such an alternative he need only earn $3,634.22 annually at a full time job in order to meet his expected income from dairy farming. This is a very inadequate economic incentive for dairy farming in view of the growing concern that the United States may face a domestic food shortage in the future. Of course, one must keep in mind that we have ignored noneconomic incentives in the foregoing analysis.

When a comparison of dairy farms with other farms was made using the data in Chart 1 plus additional data, the Dairy Marketing Advisory Committee found that of the 13 farm enterprises showing lower returns per $100 invested (after family labor was deducted from gross returns at hired labor rates) only three required as much total capital per farm as Grade A milk production. These were hog - beef fattening in the Corn Belt and cattle and sheep ranches in the Southwest.

The Associated Dairymen study also points out that the milk producer had not shared in the increasing prosperity of the economy as of 1965. This was again true in 1966. Although there had been an increase in farm prices during 1966, the dairy farmer did not share in this increase to any significant amount. Most of the six percent increase in food costs which could be traced to prices at the farm level was associated with increases in the prices of meats,

Type of farm and location[1]	Size of farm		Labor used		Farm capital, Jan. 1					Gross farm income[2]	Total farm operating expenses[3]	Net farm income	Charge for capital[4]	Return to operator and family labor, per hour
	Unit	Number	Total	Operator and family	Land and buildings	Machinery and equipment	Livestock	Crops	Total					
			Hours	*Hours*	*Dollars*	*Dollars*	*Dollars*	*Dollars*	*Dollars*	*Dollars*	*Dollars*	*Dollars*	*Dollars*	*Dollars*
Dairy farms:														
Central Northeast	Milk cows	33.0	4,610	3,800	25,100	8,200	9,200	3,000	45,500	14,937	10,759	4,178	1,866	0.61
Eastern Wisconsin:														
Grade A	do	33.3	4,750	4,270	41,550	12,820	11,360	6,190	71,950	16,906	10,365	6,511	2,950	.84
Grade B	do	22.1	3,870	3,780	30,900	6,480	6,020	4,790	48,190	9,773	6,441	3,332	1,976	.36
Western Wisconsin, Grade B	do	21.8	4,390	4,100	22,120	4,390	7,280	4,980	38,770	8,673	5,836	2,837	1,590	.30
Dairy-hog farms, Southeastern Minnesota	do	22.2	4,240	3,910	39,400	7,550	6,620	3,940	57,600	11,101	7,200	3,901	2,362	.39
Egg-producing farms, New Jersey	Layers	5,100	4,900	4,020	35,960	2,260	7,210	0	45,430	27,439	24,969	2,470	1,863	.15
Broiler farms:														
Maine	Number	67,900	2,330	1,960	24,340	8,820	0	0	33,160	7,651	3,959	3,692	1,360	1.19
Delmarva:														
Broilers	do	53,237	1,840	1,600	17,320	2,770	0	0	20,090	4,702	2,269	2,433	824	1.01
Broiler-crop	do	59,590	2,650	2,370	40,900	10,420	0	0	51,320	10,437	4,415	6,022	2,151	1.63
Georgia	do	28,314	1,509	1,430	12,610	4,030	730	160	17,580	2,608	1,890	718	721	(5)
Corn Belt farms:														
Hog-dairy	Cropland	125	4,620	3,600	53,020	8,310	7,930	5,280	74,540	18,258	11,085	7,173	3,056	1.14
Hog fattening—beef raising	do	138	3,710	3,310	44,350	6,650	7,160	4,490	62,650	12,222	7,827	4,395	2,738	.59
Hog-beef fattening	do	195	3,910	3,200	86,130	10,470	15,640	11,450	123,720	34,356	25,713	8,643	5,470	.99
Cash grain	do	233	3,240	2,920	128,450	8,810	1,780	1,800	140,870	22,652	10,447	12,205	5,950	2.13
Cotton farms:														
Southern Piedmont	do	101	5,070	2,350	28,950	2,210	1,050	540	32,750	7,639	4,365	3,274	1,462	.77
Mississippi Delta:														
Small	do	40	2,720	1,920	13,200	3,380	500	170	17,250	5,226	2,843	2,383	768	.84
Large-scale	do	610	25,170	3,200	236,000	40,890	7,860	1,870	286,620	70,684	45,061	25,623	13,047	---------
Texas:														
Black Prairie	do	240	2,830	2,320	52,500	7,020	2,070	580	62,170	10,383	5,715	4,668	2,066	.86
High Plains (nonirrigated)	do	461	3,050	1,850	82,500	10,590	720	360	94,170	8,905	7,229	1,676	3,907	−1.25
High Plains (irrigated)	do	413	5,520	2,450	132,560	17,460	740	400	151,520	29,784	16,881	12,903	6,697	2.57
San Joaquin Valley, Calif. (irrigated):														
Cotton-specialty crop	do	335	12,530	2,600	280,000	27,270	0	0	307,270	138,909	80,619	58,290	14,410	---------
Cotton-general crop (medium-sized)	do	335	9,790	2,600	276,250	28,540	0	0	304,790	83,937	47,870	36,067	13,981	---------
Cotton general crop (large)	do	1,178	29,540	2,690	968,510	76,500	0	0	1,045,010	253,189	151,404	101,785	47,651	---------
Peanut-cotton farms, Southern Coastal Plains	do	71	3,450	2,420	19,360	3,430	1,770	880	25,410	9,857	4,676	5,181	1,158	1.65

Chart 1. Commercial farms: costs and returns, by type and location, 1964.

Tobacco farms:														
North Carolina Coastal Plain:														
Tobacco	do	47	5,700	2,420	35,900	4,480	460	530	41,370	12,835	6,406	6,429	1,863	1.89
Tobacco-cotton	do	53	6,690	2,600	39,270	4,740	460	450	44,920	13,613	7,251	6,362	2,030	1.67
Kentucky Bluegrass:														
Tobacco-livestock, inner area	do	62	4,440	2,660	96,350	5,490	7,610	2,180	111,630	15,521	8,991	6,530	4,721	.68
Tobacco-dairy, intermediate area	do	25	3,410	3,220	18,330	3,020	2,740	960	25,050	5,673	2,995	2,678	1,101	.49
Tobacco-dairy, outer area	do	42	4,920	4,180	36,450	6,140	4,780	1,910	49,280	11,899	6,576	5,323	2,181	.75
Spring wheat farms:														
Northern Plains:														
Wheat-small grain-livestock	do	597	2,550	2,240	43,100	11,270	4,310	1,860	60,540	14,673	5,983	8,690	2,620	2.71
Wheat-corn-livestock	do	397	3,460	3,320	42,850	8,470	9,280	2,480	63,080	10,367	5,305	5,062	2,671	.72
Wheat-fallow	do	656	2,770	2,590	50,080	8,020	4,190	1,520	63,810	13,192	5,625	7,567	2,742	1.86
Winter wheat farms:														
Southern Plains:														
Wheat	do	616	2,850	2,530	91,840	11,410	7,470	2,770	113,490	14,627	6,356	8,271	4,745	1.40
Wheat-grain sorghum	do	693	2,690	2,250	111,480	10,910	9,350	1,870	133,610	15,218	8,269	6,949	5,576	.61
Pacific Northwest:														
Wheat-pea	do	551	3,630	3,040	176,660	23,620	2,280	930	203,490	28,554	13,364	15,190	8,629	2.16
Wheat-fallow	do	1,066	3,740	3,240	138,700	20,240	4,740	1,220	164,900	24,777	10,941	13,836	6,993	2.11
Cattle ranches:														
Northern Plains	Cows	101.6	4,290	3,300	53,950	7,700	25,300	3,420	90,370	13,579	7,536	6,043	3,705	.71
Intermountain Region	do	143.5	5,140	4,000	41,710	6,810	39,210	4,570	92,330	14,300	7,440	6,860	3,786	.77
Southwest	do	159.8	3,830	2,300	149,410	5,640	31,000	2,150	188,200	13,398	12,088	1,310	7,716	−2.79
Sheep ranches:														
Northern Plains	Sheep	1,392	6,800	3,540	72,530	6,930	23,810	1,170	104,440	25,567	13,802	11,765	4,282	2.11
Utah-Nevada	do	2,217	7,790	3,700	102,390	6,790	50,540	1,880	161,510	38,803	24,172	14,631	6,622	2.16
Southwest	do	1,219	5,210	2,700	195,360	5,080	21,440	960	222,840	17,109	13,851	3,258	9,136	−2.18

[1] All except cotton farms in California and large-scale cotton farms in the Mississippi Delta are family-operated.

[2] Total sales plus Government payments, perquisites, and change in inventory of livestock and crops.

[3] Includes all cash production expenditures plus depreciation and change in inventory of service buildings, machinery and equipment.

[4] Total value of farm capital, Jan. 1 times 4.1 percent, the average interest rate (1915-58) charged by Federal Land Banks on farm mortgage loans outstanding plus production credit times short-term interest rates charged by production credit associations on loans outstanding.

[5] Less than one.

Economic Research Service.

Chart 1. (continued)

fruits, and vegetables, not increases in milk and milk product prices (33). This lack of response of milk prices to an economy marked by prosperity and inflation prompted the National Farmers Organization to vote in favor of a holding action on milk in December of 1966 in an attempt to obtain an increase of two cents per quart in the price of fluid milk.

Finally, it is very worthwhile to note that the Dairy Marketing Advisory Committee concluded that the price which is necessary to guarantee a safe and adequate supply of milk in the short run has ceased to be a practicable criteria for determining an acceptable level for the price of milk because the present price level does not return equitable incomes to milk producers. This criteria ignores the concept of parity (parity is a measure of the prices required to give farmers the purchasing power for things they normally buy equivalent to the purchasing power experienced during the base period of time when economic conditions were acceptably in balance), and causes the welfare of milk producers to lag behind the growth and general prosperity of the economy.

B. Basic Concepts

The optimal marketing approach considered here to increasing dairy farm income is based on the fact that milk has several different uses. The uses of milk as defined for this study are fluid milk and cream, ice cream, evaporated milk, cheese, butter and other. The order that the different milk products are listed in also signifies the scale of prices for the milk equivalent in them, fluid milk being at the top of the scale. Since these different uses of milk can be considered as separate markets, the cash receipts of dairy farmers can be maximized by controlling the amounts of milk offered to each market. This method of analysis, which was suggested by Ladd and Kuang (27), is analagous to a price-discriminating monopolist which has more than one outlet for its product. In this study, the revenue maximizer adjusts the amount offered each outlet in such a way that returns are maximized.

Two basic optimal marketings problems are studied. The first involves maximizing cash receipts of producers when both the total quantity of milk and the quantity of milk allocated to fluid milk and the different manufactured milk products can be varied. The second problem involves maximizing cash receipts of producers when the total quantity of milk is fixed and allocation among uses can be varied. Several variations of these two basic problems arise when welfare implications are considered. These welfare implications are treated by means of alternative welfare constraints which are built on the ideas generated by the review of welfare economics in Chapter two. All problems are solved using the quadratic programming technique.

II. REVIEW OF LITERATURE

A. Related Studies

1. Optimal use of milk in the Netherlands

Because of the surplus milk problem in the Netherlands, Louwes *et al.* (29) performed a study in which they used quadratic programming in an attempt to find a suitable remedy. This surplus problem arose out of the facts that; (1) the government guarantees a price to the dairy farmer, who must turn over to a central organization of the government all of the milk which he does not use, and (2) they do not set production quotas for the farmers. In recent years, a low market price has forced the government to turn over large subsidies in order to meet the price which they guarantee the farmer.

The central organization can be looked upon as a monopolist whose price setting power is subject to government approval. Therefore Louwes *et al.* sought those prices, the instrument variables, which would maximize the revenue obtained from milk and the different milk products, thereby minimizing government subsidy expenditures, by determining how the available quantity of milk should be allocated among fat cheese, 40-plus cheese, butter, and milk for fluid consumption. To insure against obtaining socially unacceptable prices a welfare constraint was added.

The problem was set up in quadratic programming style by maximizing $\sum_{i=1}^{4} p_i x_i$ where linear demand functions of the form $x_i = f(p_1, p_2, p_3, p_4)$, i=1, 2, 3, 4, were used, and in which the x's are the quantities of each of the four products and the p's are the market prices for each product. Formulation of the problem was completed by adding product manufacturing and feasibility constraints in addition to the welfare constraint.

A feasible solution that did away with all but a small part of the government subsidy was found without using the welfare constraint, but the result was deemed highly unacceptable. The reason for this was that the result implied a doubling of the market milk prices.

The welfare function, which was added before again solving the quadratic programming problem, was constructed by attaching weights to the deviations of the product prices from prices in a particular year. The ratio of these weights for the function was equal to the ratio of the budget shares with the exception of butter, whose budget share was adjusted downward to account for the substitution of margarine. (These weights were scaled so that they summed to 10.) Also the deviations of the product prices from prices in a particular year, in this case 1960, were divided by the prices for that particular year. Thus the final form of the welfare function was

$$I = w_1 \frac{p_1 - \overline{p}_1}{\overline{p}_1} + w_2 \frac{p_2 - \overline{p}_2}{\overline{p}_2} + w_3 \frac{p_3 - \overline{p}_3}{\overline{p}_3} + w_4 \frac{p_4 - \overline{p}_4}{\overline{p}_4},$$

where I is a price constraint index of disutility, p_i are the solution prices, $\bar{p}_i$ are the prices for 1960 (called "norm" prices) and w_i are the weights. The price constraint index allows one to select an appropriately weighted average of the relative price changes. For example, an index value of 1 means an appropriately weighted average of the relative price changes is 10 per cent. It is obvious that protests over higher prices will be increasingly heard as the index rises from zero.

Several different indices were used for the constraint in order to consider alternative welfare cases. When the problem was solved using a neutral index (an index of zero) for the welfare constraint, the result was that the actual prices were very close to their optimal values. For welfare constraint indices favoring the central organization it was found that optimality would require raising the price of milk and lowering the prices of butter and cheese.

2. Optimal beef and pork marketings

In a study which did not use quadratic programming, but which is similar in other respects to the study presented in this thesis, Ladd and Kuang (27) computed optimal marketings for the United States beef-and-pork economy where optimal marketings were defined as those marketings which maximize annual gross farm income from beef and pork. Optimal marketings were computed for the case where quarterly marketings were permitted to vary while annual marketings were fixed and for the case where both annual and quarterly marketings were allowed to vary.

Ladd and Kuang used a model of the beef-and-pork marketing sector of the economy containing 16 linear equations, made up of 8 behavioral equations (4 demand and 4 margin equations) and 8 definitions. Farm marketings of beef and pork (Q_{Bt} and Q_{Pt}) were exogenous variables for estimation purposes and instrument variables for maximization purposes, whereas farm prices for beef and pork (P_{Bt} and P_{Pt}) were treated as endogenous. From this we see that annual farm income from hogs and cattle is $\sum_{t=1}^{4} P_{Bt}Q_{Bt}+\sum_{t=1}^{4} P_{Pt}Q_{Pt} = I_B+I_P = I_{B+P}$. Annual marketings are $Q_B=\sum_{t=1}^{4} Q_{Bt}$ and $Q_P=\sum_{t=1}^{4} Q_{Pt}$.

Six problems were studied for optimization purposes by setting up possible combinations for varied and fixed quarterly and annual marketings between beef and pork. For example, one such problem involved fixed quarterly and fixed annual marketings of beef with varying quarterly and fixed annual marketings of pork. The solution of these six problems was accomplished by methods of classical maximization with and without constraints. The problems could have been solved by quadratic programming just as the problems presented in this thesis could have been solved by classical maximization. Modification of the classical maximization method was necessary to solve the six problems because of the presence of lagged endogenous variables in the reduced-form equations for the model. The modification consisted of solving reduced-form difference equations before taking partial derivatives. After taking partials and equating to zero, solutions were obtained from systems of simultaneous equations and checked to verify that they fulfilled the second-order conditions for a maximum.

Unique solutions were obtained in each year for all the problems, but a couple of the problems involved non-feasible solutions since negative marketings were required. The results show that pork producers could increase net and gross income from hogs by adjusting production so as to increase first and forth quarter marketings and to decrease second and third quarter marketings, assuming that total annual marketings are fixed. Concerning total beef and pork production, results indicate that net and gross income from them could be substantially increased if marketings were cut back sharply.

Six problems similar to the above mentioned problems were carried out to maximize per capita consumer expenditures on beef and pork where constraints were quarterly and annual per capita consumption. Solutions to these problems show that with fixed annual marketings consumer expenditures on beef and pork are now being maximized, i.e., no change is needed in the present pattern of quarterly marketings. In addition, the solutions show that consumer expenditures on beef and pork could be increased some by reduced beef and pork marketings.

The authors also treat the welfare implication of the retail-price increases called for by several of the solutions. This implication is considered by using the idea of compensating variation in consumer incomes, i.e., the amount that income needs to be increased so that the individual consumer is on the same utility level as he was previous to the price change. As was previously mentioned, alternative welfare constraints were used to treat such implications for the problems solved in this study.

3. Optimum levels of milk production

The study contained in this thesis is on the macro-policy level, and so it may be worth while to briefly outline a micro-policy study which was made by Hoepner (20) for the benefit of dairy farmers. He discusses optimum cow numbers and optimum levels of milk production at the micro, or farm level, given the existing base excess pricing program. Optimum cow numbers and optimum milk production are defined as that number of cows and that level of production that will maximize profits for the individual dairy farmer.

Hoepner shows that instead of equating marginal cost to the price of surplus milk and purchasing additional base if the initial cost can be recovered in a reasonable period of time, dairymen should take into account; (1) the effect on marginal cost of production that "write off" (investment recovery period) length has, and (2) the modification of surplus production's marginal revenue so as to take account of the discounted present value of additional base when deciding what should be the optimum cow numbers and levels of production under marketing quotas. Hoepner says that optimum levels under this decision rule could greatly exceed those obtained when using the decision rule of equating marginal cost with the price of surplus milk.

The method of solution used in this analysis first involved the equating of partial derivatives of marginal costs to marginal revenues. The resulting sets of simultaneous equations could then have been solved directly. However, some equations involved second degree polynomials, so an approximation procedure was employed as an alternative method of solution.

B. Welfare Economics

This section considers some ideas on welfare economics, the purposes being to point out the necessity for welfare constaints to determine what conditions these constraints must satisfy, and to decide what they should take account of in choosing between policies. To begin with, consider the discussion undertaken in the introductory chapter. Interpreting the income and investment data given in that chapter to mean that there is a need for an increase in dairy farm incomes involves an obvious value judgment concerning the distribution of income. Upon first examining this data, there are four reasonable alternatives for a policy to be followed concerning the distribution of income. First, the status quo distribution of income could be accepted. Second, an arbitrary judgment by some authority could be considered. Third, the idea of "the less inequality of income the better" could be taken as the rule to operate by. And fourth, only policies which hurt no one while raising dairy farm incomes could be considered.

The very nature of this study rejects the first alternative. The second alternative is already at work and has given judgment in favor of raising dairy farm incomes as is evidenced by the establishment of dairy market orders. However, even though these market orders have ended market chaos and guaranteed the dairy farmer proper payment for his milk, they have failed to help raise dairy farm incomes. The third policy has been accepted in this study only in the sense that given the high investment required of dairy farmers, a movement towards equality is certainly desirable if future milk needs are to be fulfilled. It is by no means advocated here that this policy would be desirable for all economic decisions. The fourth alternative would most likely be some type of producer-handler cooperation.

The above alternatives suggest a look at the present state of theoretical welfare economics. Theoretical welfare economics, according to Mishan (31, p. 5), "is that branch of study which endeavors to formulate propositions by which we may rank, on the scale of better or worse, alternative economic situations open to society." The idea has been to rank these policies (situations) without, of course, resorting to the use of value judgments such as the one used in this study concerning income distribution. Could theoretical welfare economics, in its present state, justify a policy without the use of a value judgment? Several criteria which purport to do this have been put forth during the development of welfare economics, but all of them have failed for one reason or another. For example, the Kaldor and Scitovsky criteria have failed because it can be shown that these criteria must eventually rely on value judgments to make choices between alternative welfare distributions. (The Kaldor criterion states that a change is an improvement if, in a movement from one utility point to another, those who gain value their gains more than those who lose value their losses. The Scitovsky criterion, on the other hand, states that a change is an improvement if a movement from one utility point to another satisfies the Kaldor criterion, but that a movement from this new point back to the old point does not.) Another example is the Bergson criterion. (In the Bergson criterion, an indifference map known as the "social welfare function" is constructed based on the formulation of a set of explicit value judgments. Then, that point of utility which lies on the highest indifference curve is chosen.) This criterion is not useful in real life situations because it is not known how to construct, without the use of value judgments, the social welfare function which it utilizes. As a final resort, one might consider the use of the marginal optimality rules of welfare economics. But these too would fail to help because, as Baumol (5) points out, they are not at all adequate on the question of income distribution.

Since the present state of welfare economics, which according to most economists is said to be a sad state, offers no concrete means of avoiding the use of value judgments, the viewpoint of Little must be accepted. Little (28), who talks of replacing welfare economics with good common sense, asserts that value judgments concerning the distribution of income cannot possibly be avoided, and so says that an attempt should be made to make them explicit and widely acceptable.

To make the value judgment of this study widely acceptable, it is obvious that welfare constraints for the problems must be such that they constrain the results of the problems to those that take into account the feeling of the major pressure groups concerned with the price and quantity variables of milk and milk products: the welfare constraints should limit results to those that are acceptable to the majority of the people. These constraints could best be set up by directly interviewing leaders of labor, management, consumer and farm pressure groups; members of government and their advisers; and technical experts. These people should, prior to interviewing, discuss all possibilities and, perhaps after negotiation, should arrive at decisions which would yield the proper welfare constraints. However, since this process is not feasible at the present time, the technique of "imaginary interviewing", where all available expressed views of these policymakers are considered, can be employed. In essence, the outcome of an actual interview is forecast. Imaginary interviewing was first suggested by Van Eijk and Sandee (49). In addition to being acceptable to the majority of the people, welfare constraints, as Rothenberg (37) emphasizes, must enable the comparison of most of the policy alternatives available and must be internally consistent.

Using "imaginary interviewing" and the two rules suggested by Rothenberg, what are the implications for some alternative welfare constraints? The groups of people who should be interviewed are, of course, those who could be adversely affected most when cash receipts of farmers are maximized: consumers, processors, and the federal government, and those who could be benefitted most: dairy farmers. A significant reduction in the production of one or more of the milk products would receive protests from consumers on the basis that either nutritional standards were being violated or the standard-of-living reduced by the insufficient availability of these products. Large price increases would undoubtedly be rejected by the majority of the consumers, and with an inflationary period such as exists in our economy now, the government would also strongly resist these price increases. In addition, the government might resist drastic cutbacks in evaporated milk, for example, because of the fear that this manufacturing industry would become even more concentrated due to some of the smaller manufacturers being forced to stop production first. And finally, the processors (manufacturers) themselves would, of course, resist drastic curtailments in their manufacturing activities.

It is now obvious that alternative welfare constraints should be constructed such that the following are considered: (1) minimal health standards, (2) the volume of each product in relation to increased concentration, and (3) increases in prices of milk and milk products. Health standards, concentration, and price increases will all be considered in separate welfare constraints to avoid any inconsistencies that might arise and also to give a more complete and meaningful analysis. Comparisons will be made between the results obtained when these constraints are attached to the problems one at a time. The actual numerical formulation of the welfare constraints will be given in Chapter five.

III. MODEL

A. Brandow Study

The set of dairy product farm-level demand equations used in the present study is based on slope coefficients estimated by Brandow (8). In a study which was undertaken because of the need for evaluating price and income consequences of different forms of supply control in agriculture, Brandow estimated economic relationships connecting quantities of farm products and the prices received at the retail and farm levels. These basic relationships are set forth in the form of a demand model which is intended to describe long-run rather than short-term behavior of markets. This model, which is broken down into several parts, is tied to 1955-57 average prices and quantities.

The first part of the model describes retail demand. It is from this part that slope coefficients of domestic use farm-level demand equations are derived for the second part of the model. The third part deals with export and miscellaneous demands, and from the second and third parts combined is determined a fourth part showing total demands at the farm level of marketing.

Even though estimation of the model was based on conventional statistical analysis, it was often necessary for Brandow to replace the normal statistical practice of minimizing error variances or variance ratios with a different set of criteria. These criteria included theoretical relations, good but not "best" fits to recent data, conformance with commonly observed market behavior, and technological relationships. Partial tests in situations appropriate to the model showed it to be realistic, despite predictions being subject to some margins of error.

1. Retail-level demand

a. Elasticity conditions The retail model given by Brandow for all foods is as follows:

$$
\begin{aligned}
q_1 &= a_1 + b_{11}p_1 + b_{12}p_2 + \dots + b_{1n}p_n + b_{1h}P_h + b_{1y}y + b_{1s}s \\
q_2 &= a_2 + b_{21}p_1 + b_{22}p_2 + \dots + b_{2n}p_n + b_{2h}P_h + b_{2y}y + b_{2s}s \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.1) \\
q_n &= a_n + b_{n1}p_1 + b_{n2}p_2 + \dots + b_{nn}p_n + b_{nh}P_h + b_{ny}y + b_{ns}s \\
Q_h &= a_h + b_{h1}p_1 + b_{h2}p_2 + \dots + b_{hn}p_n + b_{hh}P_h + b_{hy}y + b_{hs}s,
\end{aligned}
$$

where the q's and p's are per capita consumption and retail prices of n foods or n food groups, P_h is the index of consumer prices of goods and services other than foods, y is disposable per capita income, s is an indicator of changing tastes and preferences, and Q_h is an index of per capita consumption of nonfood goods and services. The variables are in logarithms so that the coefficients represent conventional elasticities which show the percentage changes in quantities resulting from one per cent changes in prices and income.

In accordance with the theory of consumer demand developed by Wold and Jureen (51), Brandow imposed the following five conditions on his elasticity coefficients:

Condition(1): When consumer's preferences are constant, the sum of the direct and cross price elasticities and the income elasticity is equal to zero. An illustration of this would be $b_{11} + b_{12} + \ldots + b_{1n} + b_{1h} + b_{1y} = 0$. This relation is known as the homogeneity condition.

Condition(2): Known as the symmetry relation, this condition can be written

$$b_{ij} = \frac{w_j}{w_i} (b_{ji}) - w_j (b_{iy} - b_{jy}),$$

where b_{ij} is the cross-elasticity for the quantity of i dependent on the price of j, b_{ji} is the cross-elasticity of the quantity of j dependent on the price of i, w_i and w_j are expenditures for i and j as proportions of total expenditures, and b_{iy} and b_{jy} are income elasticities.

Condition(3): The weighted column sum $w_1 b_{1k} + w_2 b_{2k} + \ldots + w_n b_{nk} + w_h b_{hk}$ is the negative of the proportion of total expenditures accounted for by commodity k, $k = 1, 2, \ldots, n$.

Condition(4): The weighted sum of the income elasticities is unity, i.e., $w_1b_{1y} + w_2b_{2y} + \ldots + w_nb_{ny} + w_hb_{hy} = 1$.

Condition(5): Assuming that nonfoods are want-independent of each food, all the b_{ih} for foods should be the same multiple of the associated b_{iy}, i.e., $b_{2h}/b_{2y} = b_{1h}/b_{1y}$, $b_{3h}/b_{3y} = b_{1h}/b_{1y}$, etc.

After establishing the direct price elasticity, the nonfood cross-elasticity and the income elasticity in each row, the cross-elasticities were determined by a computational routine based on the above five conditions. An exception to this which is of interest here is that the cross-elasticity for fluid milk and evaporated milk was determined prior to commencement of the computational routine. In addition, most of the total cross-elasticity assigned to dairy product prices was allocated to the fluid milk price.

b. Elasticity estimates   Estimates of income elasticities were taken from available time series analyses and budget studies. The cross-elasticities showing the effects of nonfood prices on consumption of individual foods were estimated at thirty-three per cent of the associated income elasticities under the assumption that each food was want-independent of nonfood goods and services. Direct price elasticities were selected after statistical measurements were reviewed in other studies and additional estimates were obtained in Brandow's study. Following is a product by product account of how direct price elasticities of dairy products were found:

Fluid milk and cream: The elasticity (-.285) for fluid milk and cream was a downward adjustment of that computed by Rojko (36) for the period 1947-54. The downward adjustment was made because of the lower relative importance of cream, which is thought to have a more elastic demand than fluid milk.

Evaporated and condensed milk: The evaporated and condensed milk price elasticity (-.30) was estimated from time series data.

Cheese: Neither the previous estimates studied by Brandow nor those he computed himself for the price elasticity of cheese were statistically significant at the 10 per cent probability level. Therefore, somewhat based on these nonsignificant estimates, Brandow decided to arbitrarily fix the price elasticity of cheese at -.70.

Ice cream: The elasticity (-.55) is a downward adjustment of that computed from consumer panel data by Quackenbush and Shaffer (34). The adjustment was made because an important part of ice cream production is consumed in recreational settings where price probably has a reduced influence.

Butter: The price elasticity (-.85) for butter was chosen after applying several sets of fats and oils elasticities satisfying the symmetry relation to data for the 1950's.

Other uses: Other uses includes less important dairy products and milk fed to calves. The direct price elasticity (-.366) assigned to these uses was slightly lower than the price elasticity for the main products collectively at the farm. Demand is in terms of farm prices for these uses.

Exports: Estimates for commercial export demand were made in terms of farm prices and rest on the assumption that shipments under P.L. 480 and similar programs (except barter deals) are independent of commercial sales. The responses of exports to prices are given in terms of slope coefficients rather than elasticities, since when net exports are small or negative, elasticities are virtually meaningless. Estimates of these coefficients were made only for evaporated and condensed milk, cheese, and butter because foreign trade in fluid milk and ice cream is relatively unimportant. They were made after examination of foreign trade data in the post-war period, especially since 1952, and were influenced by analyses of foreign market prospects.

2. Farm-level demand

A representative retail demand equation for product i,

$$Q_i^R = a_i^R + \sum_{j=1}^{n} b_{ij}^R p_j^R + b_{it}^R\ T, \qquad (3.2)$$

can be adopted from equations 3.1. In this equation Q_i^R is total quantity in farm quantity units of measurements, and T is a measure of time. The b_{ij}^R are slope coefficients rather than elasticities because the variables are in natural units instead of logarithms. They were computed by rearrangement of the formula giving the finite approximation for the price elasticity of demand at the means of the data, i.e.,

$$b_{ij}^R = b_{ij} \frac{\bar{Q}_i^R}{\bar{p}_j^R},$$

where the bar denotes the 1955-57 average. The coefficient of T, b_{it}^R, was derived from a composite time factor for trend, rising income, and increasing population.

The above changes from logarithmic to natural unit form were made to allow the computation of farm prices. Brandow defines these farm prices as

$$p_i^F = p_i^R - m_i ,$$

where m_i is the marketing margin for product i (covering all transportation, processing, and distribution from the farm to the consumer). This margin was further broken down such that

$$m_i = F_i - d_i$$

$$= k_i + v_i - d_i ,$$

F_i representing the spread, including profits, realized by marketing firms for performing marketing services, per unit of farm quantity; d_i the value of byproducts, if any, obtained in processing 1 unit of the farm product; k_i the portion of F_i which is constant per unit of product marketed; and v_i the portion of F_i varying with the price of the product. Brandow assumed that v_i amounted to 10 per cent of p_i^R and varied in the same proportion as p_i^R. In addition, he assumed that the value of byproducts was related to farm price by the equation

$$d_i = \bar{r}_i p_i^F,$$

where

$$\bar{r}_i = \frac{\bar{d}_i}{\bar{p}_i^F} \quad .$$

It is now possible to derive a representative farm demand equation for product i. Working with the above definitions,

$$p_i^F = p_i^R - k_i - v_i + d_i$$

$$= \frac{1}{1-\bar{r}_i} \; (p_i^R - k_i - v_i).$$

Now, for projection purposes, k_i is assumed to remain at its 1955-57 average level, $\bar{k}_i$. Since v_i is assumed to be 10 per cent of p_i^R,

$$p_i^F = \frac{.9}{1-\bar{r}_i}\, p_i^R - \frac{\bar{k}_i}{1-\bar{r}_i} \,.$$

Therefore,

$$p_i^R = \frac{1-\bar{r}_i}{.9}\, p_i^F + \frac{\bar{k}_i}{.9} \,. \tag{3.3}$$

Substituting 3.3 into 3.2 gives

$$Q_i^R = a_i^R + \sum_{j=1}^{n} b_{ij}^R \left(\frac{1-\bar{r}_j}{.9}\, p_j^F + \frac{\bar{k}_j}{.9} \right) + b_{it}^R\, T$$

$$= a_i^F + \sum_{j=1}^{n} b_{ij}^R \left(\frac{1-\bar{r}_j}{.9}\, p_j^F \right) + b_{it}^R\, T, \tag{3.4}$$

where a_i^F is chosen so as to satisfy equation 3.4 at the 1955-57 means of the data. The coefficient, b_{ij}^F, of p_j^F in the farm demand equation for commodity i is

$$b_{ij}^F = b_{ij} \frac{\bar{Q}_i^R}{\bar{p}_j^R} \left(\frac{1-\bar{r}_j}{.9} \right) .$$

This slope coefficient shows the effect of a one-unit change in the farm price of commodity i on the domestic consumption of this commodity. Hence, the derived equation for domestic food use farm-level demand of commodity i with variables in natural units is

$$Q_i^F = a_i^F + \sum_{j=1}^{n} b_{ij}^F p_j^F + b_{it}^F T,$$

where $Q_i^F = Q_i^R$ and $b_{it}^F = b_{it}^R$.

B. Study Model

The slope coefficients b_{ij}^F were used in this study to set up domestic use farm-level demand equations for the six uses of milk. Total farm-level demand equations for fluid milk and cream, ice cream, and other uses are equal to domestic use farm-level demand equations, whereas total farm-level demand equations for the other three products are found by adding the total domestic use demand equation to the appropriate net export demand equation for each product.

The domestic use demand equations can be written as

$$Q_{1t}^F = a_{1t} + b_{11}^F P_{1t}^F + b_{12}^F P_{2t}^F + b_{13}^F P_{3t}^F + b_{14}^F P_{4t}^F + b_{15}^F P_{5t}^F$$

$$Q_{2t}^F = a_{2t} + b_{21}^F P_{1t}^F + b_{22}^F P_{2t}^F + b_{23}^F P_{3t}^F + b_{24}^F P_{4t}^F + b_{25}^F P_{5t}^F$$

$$Q_{3t}^F = a_{3t} + b_{31}^F P_{1t}^F + b_{32}^F P_{2t}^F + b_{33}^F P_{3t}^F + b_{34}^F P_{4t}^F + b_{35}^F P_{5t}^F$$

$$Q_{4t}^F = a_{4t} + b_{41}^F P_{1t}^F + b_{42}^F P_{2t}^F + b_{43}^F P_{3t}^F + b_{44}^F P_{4t}^F + b_{45}^F P_{5t}^F$$

$$Q_{5t}^F = a_{5t} + b_{51}^F P_{1t}^F + b_{52}^F P_{2t}^F + b_{53}^F P_{3t}^F + b_{54}^F P_{4t}^F + b_{55}^F P_{5t}^F$$

$$Q_{6t}^F = a_{6t} + b_{66}^F P_{6t}^F$$

and the net export demand equations as

$$X^F_{2t} = a_{7t} + e^F_{21}p^F_{2t}$$

$$X^F_{3t} = a_{8t} + e^F_{33}p^F_{3t} \quad t = 1951, \underset{64,}{55,}\ 60, \text{ and}$$

$$X^F_{5t} = a_{9t} + e^F_{55}p^F_{5t}$$

where fluid milk and cream, evaporated and condensed milk, cheese, ice cream, butter, and other uses correspond to the subscript numbers 1,2,3,4,5, and 6 respectively, and a_i is not the same as a^F_i. Note that cross-elasticities, and therefore slope coefficients, were not computed for other uses, and that a similar situation exists for export demands. Total demand for evaporated and condensed milk, for example, would be

$$Q^F_{2t} + X^F_{2t} = (a_{2t} + a_{7t}) + b_{21}p^F_{1t} + (b_{22}+e_{22})p^F_{2t} + b_{23}p^F_{3t}$$
$$+ b_{24}p^F_{4t} + b_{25}p^F_{5t} + b_{26}p^F_{6t}.$$
$$= c_{2t} + d^*_{21}p^F_{1t} + d^*_{22}p^F_{2t} + d^*_{23}p^F_{3t} + d^*_{24}p^F_{4t} + d^*_{25}p^F_{5t} + d^*_{26}p^F_{6t}.$$

Given data for the years 1951, 55, 60, and 64, on domestic food use, net exports, and farm prices for each of the six products, the constant terms a_{it}, i=1, 2, . . . , 9; t=1951, 55, 60, 64, were determined by substituting the price and quantity data for the t-th year into the equations and solving for a_{it}. Tables 1 and 2 give the b^F_{ij}, e^F_{ij}, and a_{it} for the domestic use and net export

Table 1. Slope coefficients, b_{ij} and e_{ij}, for domestic use and export demand equations[a]

i	b_{i1}	b_{i2}	b_{i3}	b_{i4}	b_{i5}	b_{i6}	e_{i2}	e_{i3}	e_{i5}
1	-15.99	1.03	.1761	.0712	.1295				
2	1.094	-2.496	.0265	.01187	.04163		-2.4		
3	.05402	.007686	-14.27	.02373	.07136			-5.0	
4	.0317	.004583	.03537	-2.808	.0418				
5	.09884	.01442	.1101	.04639	-68.99				-1.2
6						-5.073			

[a]In explanation; if, for example, i was set equal to 2 (that is, the row subscript is fixed at 2), then we would be referring to the slope coefficients of the domestic use demand equation for evaporated milk, $b_{21} = 1.094$, $b_{22} = -2.496$, etc., and the export demand slope coefficient, $e_{22} = -2.4$, for evaporated milk.

Table 2. Constant terms, a_{it}, for domestic use and net export demand equations[a]

t	a_{1t}	a_{2t}	a_{3t}	a_{4t}	a_{5t}	a_{6t}	a_{7t}	a_{8t}	a_{9t}
1951	626.28	68.07	169.13	80.30	519.05	64.05	15.05	18.00	4.61
1955	650.49	60.82	182.93	89.86	495.99	59.06	12.18	10.45	4.48
1960	658.43	55.53	193.23	102.40	469.08	51.11	11.41	9.35	3.99
1964	661.25	50.19	216.30	110.62	466.76	51.87	10.57	7.25	3.20

[a]In explanation; if, for example, t was set equal to 1951, then we would be referring to the constant terms $a_{1,\ 1951} = 626.28$, $a_{2,\ 1951} = 68.07$, etc.

demand equations, and Tables 3 and 4 give the constant terms, c_{it}, and slope coefficients, d_{ij}, for the total demand equations.

A detailed explanation of the problems studied and how they were set up will be given in Chapter five. The following chapter is a review of the mathematical technique which will be used to solve these problems.

Table 3. Constant terms, c_{it}, for total demand equations[a]

t	c_{1t}	c_{2t}	c_{3t}	c_{4t}	c_{5t}	c_{6t}
1951	626.28	83.12	187.13	80.30	523.66	64.05
1955	650.49	73.00	193.38	89.86	500.47	59.06
1960	658.43	66.94	202.58	102.40	473.07	51.11
1964	661.25	60.76	223.55	110.62	469.96	51.87

[a]In explanation; if, for example, t was set equal to 1951, then we would be referring to the constant terms $c_{1,\ 1951} = 626.28$, $c_{1,\ 1951} = 83.12$, etc.

Table 4. Slope coefficients, d^*_{ij}, for total demand equations[a]

i	d^*_{i1}	d^*_{i2}	d^*_{i3}	d^*_{i4}	d^*_{i5}	d^*_{i6}
1	-15.99	1.03	.1761	.0712	.1295	
2	1.094	-4.896	.0265	.01187	.04163	
3	.05402	.007686	-19.27	.02373	.07136	
4	.0317	.004583	.03537	-2.808	.0418	
5	.09884	.01442	.1101	.04639	-70.19	
6						-5.073

[a]In explanation, if, for example, i was set equal to 1, $d^*_{11} = -15.99$, $d^*_{12} = 1.03$, etc.

IV. QUADRATIC PROGRAMMING

A. Similarities to Linear Programming

Quadratic programming, which is defined as the maximization of a quadratic objective function subject to linear constraints (all or at least some of which are in the form of inequalities rather than equalities), can be viewed as an extension of linear programming. In addition, many of the algorithms developed for quadratic programming are simply variations of the simplex method used for linear programming. For these reasons, linear programming will be briefly discussed before taking up the theory of quadratic programming. Matrix algebra will be employed extensively throughout this chapter, and the notation used will be that given in a standard matrix theory book such as Hadley (15) or Perlis (32).

1. Simplex method

The linear programming problem attempts to determine the n by 1 column vector $x \geq 0$ such that the linear objective function $z=cx$, where c is a 1 by n row vector of prices, is maximized subject to the linear constraints $Ax\ (\leq, =, \geq)\ b$, where A is m by n and b is m by 1. It is only necessary to study maximization problems because minimization problems can be converted to maximization by changing the signs of the prices, i.e., the minimum of f(x) equals the negative of the maximum of the negative of $f(x)$. The first step in solving this problem is to convert all inequalities to equalities. For $\leq$ we add slack variables, and for $\geq$ we substract surplus variables. The simplex method requires, first of all, a basic feasible solution. If the augmented A contains an identity matrix after the conversion of inequalities, then a basic feasible solution already exists. If there is not an identity matrix in A, then we add a number of artificial variables sufficient to yield an identity matrix in A. In order to force the values of the artificial variables to zero we assign them a price of $-M$.

Given this basic feasible solution, the simplex method maintains feasibility and seeks optimality by changing one of the basic variables at each step. It can be shown that the problem terminates in a finite number of steps. For expository purposes, consider the example

$$\text{maximize } z=cx$$

$$\text{subject to } Ax\leq b$$

$$\text{and } x\geq 0,$$

where c is 1 by n, x is n by 1, A is m by n, b is m by 1, and we add m slack variables with $c_i=0$, $i=n+1,\ldots, n+m$. Letting A^* equal the appropriately augmented A matrix, the problem has the initial simplex tableau shown in Figure 1, basic variables being denoted by x_B and nonbasic variables by x_{NB}, and z_j being equal to the cost coefficients vector of the basis times the columns of x_B and x_{NB}.

basis	levels	0 X_B	c X_{NB}
X_B	b	I	A^*
	z	$(z_j - c_j)$	row

Figure 1. Initial simplex tableau

The following brief description of the simplex method is intended only to recall the procedure and to fix notation. Further familiarity with the simplex method can be obtained by referring to a standard text such as Gass (14) or Hadley (16). If $z_j-c_j\geq 0$, $j=1,\ldots,n$, the basic feasible solution is optimal. If for a particular j, $z_j-c_j<0$, and all elements are less than or equal to zero in the column of x_{NB} (let these elements be the p_{ij}) corresponding to this z_j-c_j, then there is an unbounded solution. If we have $z_j-c_j<0$ for one or more j corresponding to columns of x_{NB} with at least one p_{ij} greater than zero, we seek optimality by changing the basis according to the following rules:

Rule(1): The vector to enter the basis, the vector k, is the one corresponding to the $\min_j(z_j-c_j)$ for $z_j-c_j<0$.

Rule(2): The vector to leave the basis, the vector r, is computed as $x_{Br}/p_{rk} = \min_i(x_{Bi}/p_{ik})$ for $p_{ik}>0$. If there is a tie, choose the vector using the perturbation technique as outlined in Gass (14).

With the exchange of variables completed, transform the simplex tableau using the elimination formulas $p'_{ij}=p_{ij}-\frac{p_{ik}}{p_{rk}}p_{rj}$ for $i\neq r$ and $p'_{rj}=p_{rj}/p_{rk}$. The procedure continues until terminated by the previously mentioned tableau characteristics.

2. Illustrations

Introduction of the relationships between linear and quadratic programming can be accomplished by presenting an illustration of a problem of each type. Consider first the problem

$$\text{Maximize } z = x_1 - x_2$$

$$\text{subject to } x_1+x_2\le 3$$
$$x_1-2x_2\le 1$$
$$-2x_1+2x_2\le 2$$
$$\text{and } \quad x_1,x_2\ge 0 \ .$$

Using the simplex method for this problem, we arrive at an optimal solution of $\overline{x}_1=7/3$, $\overline{x}_2=2/3$, and $\overline{z}=5/3$. Figure 2 shows the convex feasible set of solutions ϕ along with some alternative values of the objective function $z(z_1=8/3,\ \overline{z}=5/3,\ z_2=2/3)$. Note that the solution is an extreme point of ϕ: this is always true. For solutions, the convex feasible set ϕ will be a region in n-Euclidean space and can be either void, a convex polyhedron, or an unbounded convex solution. The first problem will have no solution, the second a solution with a finite value, the third a solution with a possible unbounded value. For a detailed explanation of convex sets and properties see Karlin (23).

Consider now the problem

$$\text{maximize } z=8x_1+4x_2-x_1^2$$
$$\text{subject to } 2x_1+3x_2\le 6$$
$$2x_1+x_2\le 4$$
$$\text{and } \quad x_1,x_2\ge 0.$$

This problem, solved by the Wolfe simplex method which will be presented later, has an optimal solution of $\overline{x}_1=3/2$, $\overline{x}_2=1$,

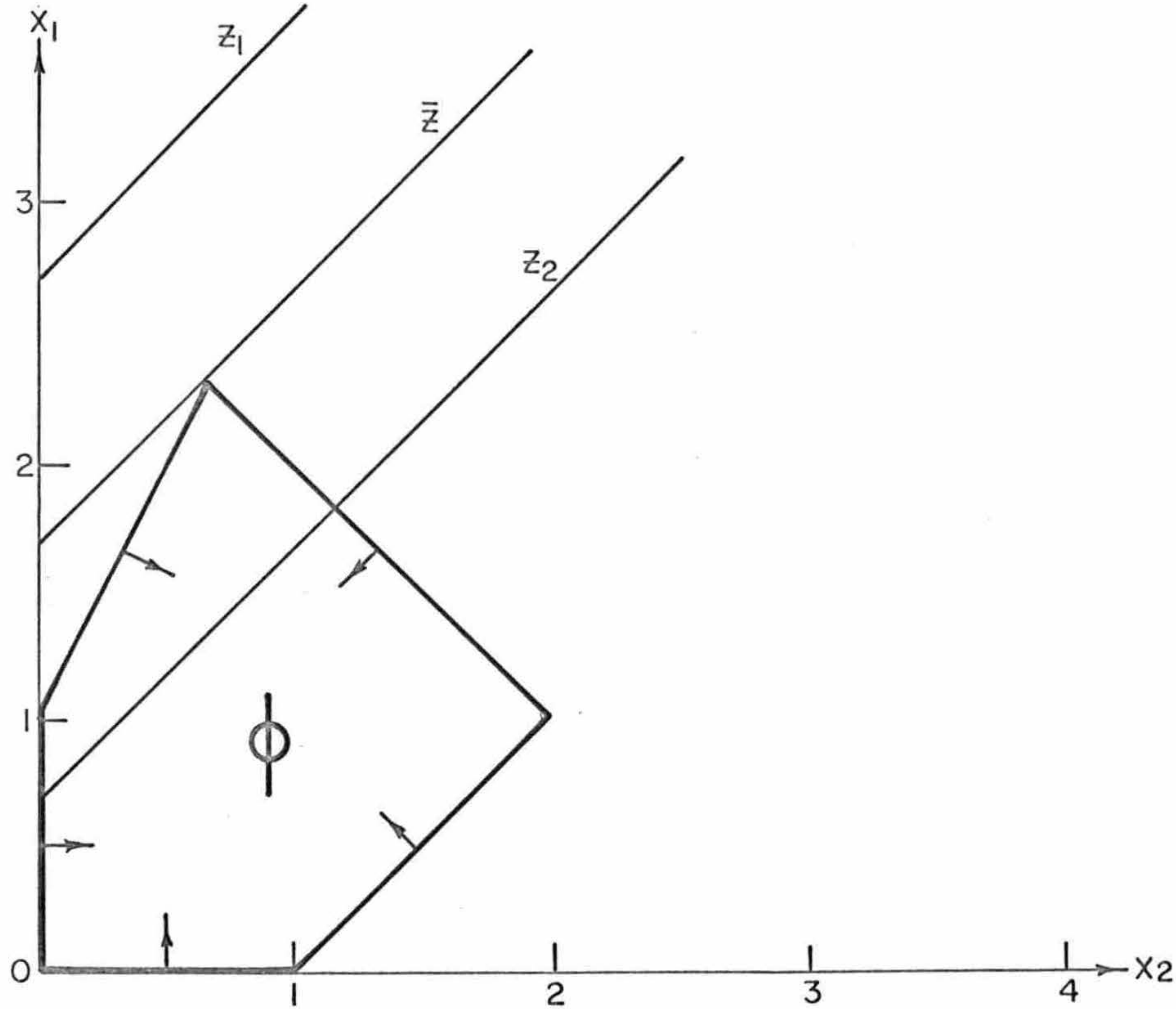

Figure 2. Linear programming problem

and $\bar{z}=55/4$. Figure 3 shows ϕ and some alternative values of the objective function $z(z_1=16,\ \bar{z}=55/4$, and $z_2=12)$. Notice that the only difference between a linear programming problem and a quadratic one is that the objective function is nonlinear in the latter.

B. Kuhn-Tucker Theory

The Kuhn-Tucker theory has been at the basis of the development of many of the algorithms for solving quadratic programming problems, and it is for this reason that the results of this theory will be covered. In order to consider the theory in its generalized form, define the gradients

$$\nabla_x F(x,\lambda) = \left[\frac{\partial}{\partial x_1} F(x,\lambda),\ldots,\frac{\partial}{\partial x_n} F(x,\lambda)\right]$$

$$\text{and } \nabla_\lambda F(x,\lambda) = \left[\frac{\partial}{\partial \lambda_1} F(x,\lambda),\ldots,\frac{\partial}{\partial \lambda_m} F(x,\lambda)\right] ,$$

where $F(x,\lambda)$ is the Lagrangian function $F(x,\lambda) = f(x) + \sum_{j=1}^{n} \lambda_i(b_i - g_i(x))$, and let J be the subset of indices for which $x_j^*>0$, where x^* represents the vector x which is an optimal solution to the nonlinear programming problem

$$\begin{aligned}
&\text{maximize } z=f(x)\\
&\text{subject to } g_i(x)\le b_i,\ i=1,\ldots,u,\\
&\qquad\qquad\quad g_i(x)\ge b_i,\ i=u+1,\ldots,v,\\
&\qquad\qquad\quad g_i(x)=b_i,\ i=v+1,\ldots,m,\\
&\qquad\quad \text{and } x\ge 0.
\end{aligned}$$

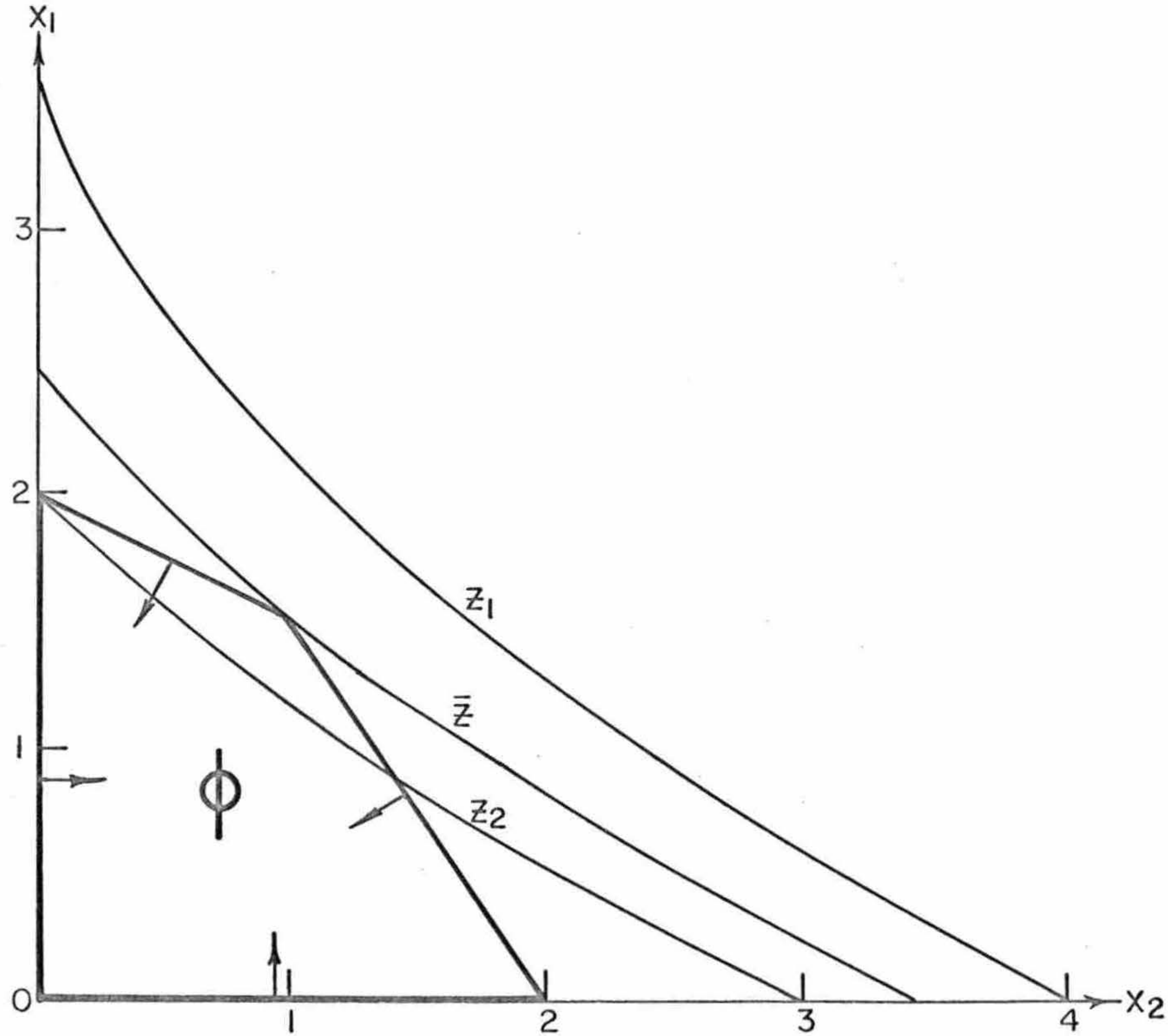

Figure 3. Quadratic programming problem

Assume that $f \in C^1$, and $g_i \in C^1$, $i=1,\ldots,m$, ($\in$ means "to be a member of") over the entire nonnegative orthant, i.e., where $x \geq 0$. A function $f(x)$ is defined as being a member of C^1 if, and only if, the first derivative of $f(x)$ belongs to the class of continuous functions. (For a discussion of gradients, Lagrangian functions, and continuous functions see Widder (50).) Upon the addition of non-negative slack and surplus variables we have the problem

$$\begin{aligned} &\text{maximize } z=f(x) \\ &\text{subject to } g_i(x)+x_{si}=b_i, \quad i=1,\ldots,u \\ &\qquad\qquad g_i(x)-x_{si}=b_i, \quad i=u+1,\ldots,v, \\ &\qquad\qquad g_i(x) \qquad\ =b_i, \quad i=v+1,\ldots,m, \\ &\qquad \text{and} \quad x \geq 0,\ x_s \geq 0. \end{aligned}$$

The Kuhn-Tucker theorem can now be stated in the following manner. For x^* to be the absolute maximum of $f(x)$ for $x \in y$, y being the set of $x \geq 0$ satisfying the constraints, it is necessary that there exist a λ^* such that the following four conditions are satisfied:

$$\text{Condition(1): } \nabla_x F(x^*, \lambda^*) = \nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) \leq 0, \tag{4.1a}$$

with the strict inequality being satisfied for $j \in J$, where $\nabla f(x^*)$ is the gradient of the objective function after the optimal solution vector has been substituted into the objective function, and $g_i(x^*)$ is the value of the i-th constraint after x^* has been substituted into this constraint.

Condition(2): $\nabla_x F(x^*, \lambda^*)\, x^* = \sum_{j=1}^{n} x_j^*$

$$\left\{ \frac{\partial f(x^*)}{\partial x_j} - \sum_{i=1}^{m} \lambda_i^* \frac{\partial g_i(x^*)}{\partial x_j} \right\} = 0. \qquad (4.1b)$$

Condition(3): The first u components of $\nabla_\lambda F(x^*, \lambda^*) = (b_1 - g_1(x^*), \ldots, b_m - g_m(x^*))$ are non-negative, while components $u+1, \ldots, v$ are non-positive, and $v+1, \ldots, m$ vanish.

Condition(4): $\nabla_\lambda F(x^*, \lambda^*)\lambda^* = \sum_{i=1}^{m} \lambda_i^* (b_i - g_i(x^*)) = 0.$ (4.1c)

If these four conditions are satisfied, then the point (x^*, λ^*) also satisfies the necessary conditions for the Lagrangian function $F(x, \lambda)$ to have a saddle point at (x^*, λ^*) for $x \geq 0$, and $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$ (where $\lambda^{(1)}$ has u components, $\lambda^{(2)}$ has v-u components, and $\lambda^{(3)}$ has m-v components) is such that $\lambda^{(1)} \geq 0, \lambda^{(2)} \leq 0$, and $\lambda^{(3)}$ is unrestricted in sign. If, in addition to the point (x^*, λ^*) satisfying the four necessary conditions for an optimal solution, $f(x)$ is concave for $x \geq 0$, $g_i(x)$ is convex when $\lambda_i^* > 0$, and $g_i(x)$ is concave when $\lambda_i^* < 0$, $i = 1, \ldots, m$, then $f(x^*)$ is the absolute maximum of $f(x)$ for $x \in y$.

To summarize, first the conditions which would insure a relative maximum for a point x^* were given, and this was said to be a global maximum if $f(x)$ is concave for $x \geq 0$ and the constraints are either convex or concave as required by the sign of λ_i^*. If the constraints are linear, then note that this constraint condition is automatically satisfied since a linear function can be thought of as either convex or concave. Given these linear constraints and that f(x) is differentiable, the necessary conditions of the Kuhn-Tucker theorem will be satisfied. For a rigorous proof of this see Kuhn and Tucker (24).

C. Preliminary Results

1. Concavity

The Kuhn-Tucker theory points out that since the desired extremum is a maximum in the quadratic programming problem being considered, the restriction that the quadratic function be concave must be imposed in order to prevent the existence of various local extrema. The quadratic objective function is the sum of a linear form and a quadratic form and is usually written as $cx+x'Dx$. Since cx can be considered concave it is only necessary to check to see that $x'Dx$ is concave in order to insure that the objective function is concave. The quadratic form $x'Dx$ will be concave if it is negative semidefinite and strictly concave if it is negative definite. Assuming that D is symmetric, the quadratic form will be negative semidefinite if, and only if, each characteristic root of D is nonpositive and at least one characteristic root is zero, and negative definite if, and only if, each characteristic root of D is negative. Alternatively, if we define a principal minor of a determinant as a subdeterminant formed by crossing out the same rows and columns so that the diagonal elements of the minor are contained in the diagonal elements of the original matrices, then the necessary and sufficient conditions for a matrix to be negative definite are that the first principal minor is negative, and the successive ones alternate in sign, those of odd order being negative, and those of even order positive. If the objective function is strictly concave, then the global maximum is unique. For a further discussion of negative semidefinite and negative definite quadratic forms and concavity see Boot (7) or Vajda (46).

2. Reformulation of problem

Having at hand a method of testing for the concavity of the objective function, it is possible to employ the Kuhn-Tucker conditions in order to reformulate the general quadratic programming problem. Assuming that all constraint inequalities have been converted to equalities by means of slack and surplus variables, the general problem can be written

$$\text{maximize } z=cx+x'Dx$$

$$\text{subject to } Ax=b$$

$$\text{and } x\geq 0,$$

where A is m by n, D is n by n, b is m by 1, c is 1 by n and x is n by 1. The matrix D will be taken as symmetric. This can be done without loss of generality. To prove this, consider $x'Bx$ and note that when $i\neq j$, the coefficient of

$x_i x_j$ is $a_{ij}+a_{ji}$. Therefore B can always be considered symmetric, because when it is not it is possible to uniquely define new coefficients

$$d_{ij}=d_{ji}=\frac{b_{ij}+b_{ji}}{2}, \text{ for all i and j,}$$

so that $d_{ij}+d_{ji}=b_{ij}+b_{ji}$ and $D=\|d_{ij}\|$ is symmetric. This redefinition does not change the value of z for any solution vector x. Besides D being symmetric, $x'Dx$ will be assumed to be negative semidefinite or definite.

For the above defined quadratic programming problem the functions $g_i(x)$ and $f(x)$ as employed in the discussion of the Kuhn-Tucker theory are $g_i(x)=a^i x$ and $f(x)=cx+x'Dx$, a^i representing the ith row of A. Reformulation of the original problem is accomplished in the following manner. Taking partial derivatives with respect to x_j,

$$\frac{\partial g_i}{\partial x_j} = a_{ij} \ ; \nabla g_i(x) = a^i, \tag{4.2}$$

and

$$\frac{\partial f}{\partial x_j} = c_j + 2\sum_{i=1}^{n} x_i d_{ij}; \nabla f(x) = c + 2x'D. \tag{4.3}$$

The four conditions of the Kuhn-Tucker theorem must be satisfied for x^*, λ^* if $x^* \geq 0$ is an optimal solution. Substituting 4.2 and 4.3 into 4.1a, the result is

$$c + 2(x^*)'D - (\lambda^*)' A \leq 0. \tag{4.4}$$

Transposing 4.4,

$$c' + 2Dx^* - A'\lambda^* \leq 0$$

is obtained. Now introduce a slack variable vector x_s^*, thereby giving

$$c' + 2Dx^* - A'\lambda^* + x_s^* = 0. \tag{4.5}$$

The second Kuhn-Tucker condition can then be written as

$$(x^*)'x_s^*=0. \tag{4.6}$$

For condition three the corresponding requirement for the optimal solution to the quadratic programming problem is

$$Ax^*=b. \tag{4.7}$$

Condition four causes no problem because this condition will be satisfied by any feasible solution-it is not necessary to include this condition specifically.

It is now apparent that given the specifications of the objective function that are assumed throughout this exposition, the application of the Kuhn-Tucker conditions to the problem shows that $x^* \geq 0$ is an optimal solution if there is a λ^* and a $x_s^* \geq 0$ such that 4.5, 4.6, and 4.7 are satisfied. That is, since the linear constraints can be considered either convex or concave, and $f(x)$ is assumed concave, the Kuhn-Tucker conditions become sufficient for an optimal solution. Finding an optimal solution to a quadratic programming problem reduces to the problem of finding $x \geq 0$, $x_s \geq 0$, λ which satisfy

$$\begin{aligned} &Ax=b \\ &2Dx-A'\lambda+x_s=-c' \\ &x'x_s=0. \end{aligned} \tag{4.8}$$

D. Wolfe Algorithm with Charnes Modification

Philip Wolfe (52) developed an algorithm to find $x \geq 0$, $x_s \geq 0$, and λ satisfying equations 4.8 when $x'Dx$ is negative definite. (The case where $x'Dx$ is negative semidefinite will be treated later.) Although the algorithm is not a straightforward generalization of the simplex method for linear programming, it is the best known procedure and the simplest to program since it requires little more than linear programming computer codes. In deriving the algorithm, first of all note that if there exists a solution (x, λ, x_s) for

$$\begin{bmatrix} A & 0 & 0 \\ 2D & -A' & I_n \end{bmatrix} \begin{bmatrix} x \\ \lambda \\ x_s \end{bmatrix} = \begin{bmatrix} b \\ -c' \end{bmatrix} \qquad (4.9)$$

where $x \geq 0$, $x_s \geq 0$, and $x'x_s=0$, then the number of non-zero components of (x, λ, x_s) cannot exceed the number of equations $(m+n)$ in 4.9. The reason for this is that λ has m components, and, since $x'x_s=0$, the number of nonzero components that x and x_s have in sum total cannot exceed n. Therefore a solution (x, λ, x_s) to 4.8 with $x \geq 0$ and $x_s \geq 0$ is a basic solution to 4.9. The important conclusion is that if a solution to 4.9 exists only basic solutions to 4.9 need be considered in order to solve 4.8. (This result was originally derived by Barankin and Dorfman (2).) Another very useful result, which is proved by Hadley (17), is that when $x'Dx$ is negative definite there can be no unbounded solution: if a feasible solution exists, then a unique optimal feasible basic solution exists.

To find the basic solution to equations 4.9 which will give the optimal solution to the original quadratic programming problem, a computational method employing a modified technique of artificial variables is used. First, using the fundamental simplex method, find a basic feasible solution to $Ax=b$ providing such a solution exists. If a feasible solution does exist, then it is known that there exists a solution to 4.9 with $x\geq 0$, $x_s\geq 0$, $x'x_s=0$. In finding this solution, let P be the basis matrix and x_p the basis variables vector so that $Px_p=b$. Since the first m constraints of 4.9 are now satisfied, it is only necessary to add the vector of artificial variables as follows:

$$\begin{aligned} &Ax=b \\ &2Dx-A'\lambda+x_s+Ex_a=-c' \end{aligned} \qquad (4.10)$$

where $x_a\geq 0$ and E is a diagonal matrix with elements

$$e_j = \begin{cases} +1 \text{ if } -c_j-2d_p^j x_p \geq 0 \\ -1 \text{ if } -c_j-2d_p^j x_p < 0 \end{cases}$$

where D_p consists of the columns of D which correspond to the columns of A in P and j denotes a row subscript. Now set $x_{aj}=\left|-c_j-2d_p^j x_p\right|\geq 0$, $j=1,\ldots,m$, $\lambda=0$, $x_s=0$, to get a basic feasible solution to equations 4.10 that contains n+m positive variables or less. This solution is written

$$\begin{bmatrix} P & 0 \\ 2D_p & E \end{bmatrix} \begin{bmatrix} x_p \\ x_a \end{bmatrix} = \begin{bmatrix} b \\ -c' \end{bmatrix},$$

the solution being basic because the matrix of coefficients is nonsingular. The inverse of the coefficients matrix as determined by the partitioning method for computing the inverse as given in Faddeeva (12) or Ralston and Wilf (35) is

$$\begin{bmatrix} P^{-1} & 0 \\ -2ED_pP^{-1} & E \end{bmatrix}.$$

With this basic solution at hand, maximize the negative of the sum of the artificial variables with $x'x_s=0$. When the sum of the artificial variables is zero the optimal solution to the quadratic programming problem has been determined. This method amounts to solving the nonlinear programming problem

$$\text{maximize } z=-\sum_j x_{aj}$$

$$\text{subject to} \begin{bmatrix} A & 0 & 0 & 0 & 0 \\ 2D & -A' & A' & I_n & E \end{bmatrix} \begin{bmatrix} x \\ r \\ s \\ x_s \\ x_a \end{bmatrix} = \begin{bmatrix} b \\ -c' \end{bmatrix},$$

$$x'x_s=0,$$

$$\text{and } x\geq 0,\ r\geq 0,\ s\geq 0,\ x_s\geq 0,\ x_a\geq 0,$$

where for all variables to be nonnegative in the new programming problem $\lambda = r-s$. Note that the only nonlinear part of the problem is $x'x_s=0$. The problem is solved using the simplex method with the exception that if $x_j>0$, x_{sj} is not allowed to enter the basis and vice versa. More generally, allow x_j and x_{sj} in the basis, but only if $x_jx_{sj}=0$. It is easily noted that even the most trivial quadratic programming problems lead to sizeable problems to which the simplex method must be applied. Wolfe shows that the problem can be solved in a finite number of iterations-the method does terminate. If the problem of degeneracy arises, it can easily be resolved by using one of the standard linear programming procedures for this problem, such as the perturbation technique. Since only the case where $x'Dx$ is negative definite has been solved, the next undertaking is to solve the problem when $x'Dx$ is negative semidefinite.

Wolfe has also supplied a technique for solution in the negative semidefinite case, but his technique is very complex and has not been used extensively. A much more widely adopted technique is the one developed by Charnes (9). He suggests a very simple modification of the above program utilizing the fact that a negative semidefinite form $x'Dx$ can be converted into a negative definite form by making an arbitrarily small change in the diagonal elements of D. That is, if $x'Dx$ is negative semidefinite, then $x'(D+\delta I)x$ is negative definite for any $\delta<0$ however small $|\delta|$. This can easily be proved since it is known that $x'Dx\leq 0$ for any x and $\delta x'Ix<0$ for any $x\neq 0$, and therefore $x'(D+\delta I)x<0$

for any $x \neq 0$. The latter is the definition of a negative definite form. From this it follows that $x'Dx$ can be made negative definite by subtracting a unit in the fourth or fifth decimal place of each diagonal element. Doing this, the perturbation will be made small enough that it will not affect the numerical results of the problem.

The Wolfe method with the modification by Charnes has been more widely used than any other quadratic programming algorithm. This is due to the aforementioned fact that very little modification of the simplex code for linear programming on a computer is needed in order to turn it into a code for quadratic programming. However, the algorithm recently developed by Van de Panne and Whinston has been proven to be more efficient than the Wolfe algorithm, and should provide stiff competition in the future. The facts that the Van de Panne and Whinston algorithm is more difficult to program and that the Wolfe algorithm requires only modification of existing programs will, of course, limit the use of the new algorithm. The Van de Panne and Whinston algorithm, recently programmed by Janet J. Zrubek (53) for research use at Iowa State University, will be discussed in detail next. For a look at various other methods, some of them applicable only to special types of quadratic programming problems, see Barankin and Dorfman (1), Beale (3 and 4), Boot (7), Hartley (18), Hildreth (19), Houthakker (21), Jagannathan (22), Markowitz (30), and

Frank and Wolfe (13).

E. Whinston and Van de Panne Algorithm

1. Foundation of algorithm

Dantzig (11) recently proposed a method for quadratic programming called "a variant of the Wolfe-Markowitz algorithms" which proved to be the true generalization of Dantzig's simplex method for linear programming. Whinston and Van de Panne (48) later, but independently, presented an argument equivalent to Dantzig's, and outlined a procedure for programming the algorithm on a computer (47). This algorithm is known as the simplex method for quadratic programming.

Whinston and Van de Panne (47) have presented the general theory of their algorithm at length, and so it is not necessary to give the complete development of it here. However, it is interesting to note the foundation upon which the method rests. Rearrange the second equation of 4.8 from

$$2Dx-A'\lambda+x_s=-c'$$

$$\text{to} \quad c'+2Dx-A'\lambda=-x_s. \tag{4.11}$$

If given a solution $(x^\circ, \lambda^\circ, x_s^\circ)$ of 4.11 with some x_s° negative, say x_{sk}°, then

$$\frac{\partial F(x,\lambda)}{\partial x_k^\circ} > 0,$$

recalling that $F(x,\lambda)$ is the Lagrangian $c'x+x'Dx-\lambda(Ax-b)$.

Hence if x_k^* is increased, the value of $F(x,\lambda)$ will increase. However, it must be remembered that changes need to be made in the other x-variables in order to continue satisfying $Ax=b$. If these changes can be made where these other x-variables correspond to zero x_s-variables such that the x_s-variables remain zero, then the objective function $f(x)$ increases when x_k^* is increased. This is the iterative process of the simplex method for quadratic programming by which a nondecreasing increase in $f(x)$ takes place. The above interpretation of the Lagrangian led to the discovery of the simplex method for quadratic programming.

2. Selection of pivotal element

The algorithm being discussed is very similar to the simplex algorithm for linear programming, the difference lying in the selection of pivotal elements. In order to present the procedural rules for selecting the pivot element in the quadratic programming algorithm, the general quadratic programming problem will be written, in contrast to the simplified notation used earlier,

$$\text{maximize } z=c'x_{.}+x_{.}'Dx_{.}$$
$$\text{subject to } Ax_{.}+x_{.s}=b$$
$$\text{and } x_{.}\geq 0,\ x_{.s}\geq 0,$$

where $x_{.s}$ is an m by 1 vector of slack variables. The Kuhn-Tucker conditions, necessary and sufficient for a saddle value since we are again assuming that D is a negative semidefinite or negative definite matrix, are

$$x_s+2Dx.-A'\lambda=-c'$$

$$Ax.+x._s=b$$

$$x_s' x.+\lambda'x._s=0$$

$$x.\geq 0, \lambda\geq 0, \; x_s\geq 0, \; x._s\geq 0.$$

The initial tableau for the algorithm is shown in Figure 4.

basic variables	variable levels	x_s	λ	$x.$	$x._s$
x_s	$-c'$	I	$-A'$	2D	0
$x._s$	b	0	0	A	I

Figure 4. Initial tableau for quadratic algorithm

Notice that this initial basic tableau has the solution

$$x._s = b$$

$$x_s = -c'$$

which satisfies every Kuhn-Tucker condition except for $x_s\geq 0$. This condition would be met if c' is a non-positive vector, and 4.12 would then be an optimal solution.

Before giving the rules for selecting the pivotal element, the following definition is needed. The tableau is defined to be in standard form if

$$x_s'x.=0$$

$$\text{and } \lambda'x._s= 0 \qquad (4.13)$$

are satisfied. Rules for selection of the pivot element are as follows:

Rule(1): If the tableau is in standard form, for the variable to be brought in choose $x_{.i}$ or $x_{.sj}$ with the most negative counterpart x_{si} or λj. If the tableau is in non-standard form, i.e., equations 4.13 are not satisfied, then the procedure quarantees that there will exist two pair $(\overset{X}{x_{.i}}, \overset{U}{x_{si}})$ and $(\overset{Y}{x_{.sj}}, \overset{V}{\lambda j})$ such that one pair is basic and the other nonbasic. In this case, choose the x_{si} or λj of nonbasic pair as the variable to enter basis.

Rule(2): To choose the variable to leave the basis, let $(\overset{U}{x'_s}, \overset{V}{\lambda'}, \overset{X}{x'_{.}}, \overset{Y}{x'_{.s}})$ correspond to basic variables and $(x''_s, \lambda'', x''_{.}, x''_{.s})$ to nonbasic variables. The form of the tableau here is of no concern because the choice of the variable to leave the basis does not depend on whether the tableau is in standard or nonstandard form. Let $w^{\bullet} = \begin{bmatrix} x'_{.} \\ x'_{.s} \\ \beta \end{bmatrix} \begin{matrix} X \\ Y \\ \\ \end{matrix}$

$$\text{where } \beta = \begin{cases} \overset{U}{x_{sk}} \text{ if } \overset{X}{x_{.k}} \text{ is to enter the basis or is in the basis.} \\ \overset{V}{\lambda_k} \text{ if } x_{.sk} \text{ is to enter the basis or is in the basis.} \end{cases}$$

Let $r^{\bullet}_k$ be the vector selected by rule one. The vector to leave the basis, $w^{\bullet}_h$, is determined by calculating

$$\frac{w^{\bullet}_h}{r^{\bullet}_{hk}} = \min_i \frac{w^{\bullet}_i}{r^{\bullet}_{ik}} > 0, \text{ for } r^{\bullet}_{ik} \neq 0.$$

If there is a tie involving β, choose β. If tie does not involve β, choose ratio with largest $r^{\bullet}_{ik}$.

These two rules are used by phase II of the algorithm, which maintains feasibility and seeks optimality, the infeasibilities having been removed by phase I of the algorithm.

Once the vector to enter the basis and the vector to leave the basis have been selected, the product form of the inverse method, which is used in the revised simplex method, is employed. The revised simplex method alone has two basic computational advantages when compared to the original simplex method. The first is that for the revised method the number of computations for problems which involve a large number of zero elements in the coefficients matrix is reduced due to advantages in computer coding and memory. The second advantage is that since the revised method does not deal with the whole tableau, the amount of new information the computer must record is less. Since the product form of the inverse method reduces the number of calculations even further, it requires even less recording time than the revised simplex method. In addition, the product form of the inverse method is able to handle larger problems than the revised simplex method.

V. PROBLEMS STUDIED

Sixteen problems were analyzed, each problem being solved for the years 1951, 55, 60, and 64. These four particular years were chosen for reasons of symmetry and data availability. Data were not yet available for 1965, and farm price data for ice cream farm equivalent were not available prior to 1951. Solutions were obtained for farm prices in each problem, and these prices were then substituted into the domestic use and export demand equations in order to determine the optimal quantities, which are the instrument variables, for each product. Price data (43, 44, and 45), which is adjusted by the consumer price index, is expressed as dollars per hundredweight for each product and quantity data (43, 44, and 45) is expressed as million hundredweight in terms of milk equivalent. This was also the case for price and quantity data used by Brandow. Actual farm prices and quantities for each of the six products and actual total cash receipts of dairy farmers are given in Chapter six and the Appendix.

A. Objective Function

The objective function for each of the problems is

$$R_t^F = P_{1t}^F Q_{1t}^F + P_{2t}^F(Q_{2t}^F + X_{2t}^F) + P_{3t}^F(Q_{3t}^F + X_{3t}^F) + P_{4t}^F Q_{4t}^F +$$

$$P_{5t}^F(Q_{5t}^F + X_{5t}^F) + P_{6t}^F Q_{6t}^F.$$

Now, each of the Q^{F}'s and X^{F}'s is expressed in terms of one or more of the p^{F}'s in Chapter three, hence R^{F} is a quadratic function in the p^{F}'s. Recall that this function can be written in the form

$$R_t^F = c_t' p_t^F + p_t^{F'} D p_t^F, \; t = 1951, \; 55, \; 60, \text{ and } 64,$$

where $p_t^{F'} = (p_{1t}^F, p_{2t}^F, p_{3t}^F, p_{4t}^F, p_{5t}^F, p_{6t}^F)$, c_t' is simply the vector of constants, $(c_{1t}, c_{2t}, c_{3t}, c_{4t}, c_{5t}, c_{6t})$, in the total demand equations for the t-th year, and the symmetric matrix D is determined by the rule in Chapter four: that is, D is formed by deriving a symmetric matrix from the elements, d^*_{ij}. The matrix D is given in Table 5. Using the principal minors theorem given in Chapter four, it is easy to verify that this matrix is negative definite. The values of the determinants of the principal minors of D are -15.99; 77.16; -1486.78; 4174.53; -293,000.90; and 1,486,393.00.

Since each of the problems has the same objective function to be maximized, the only difference between problems is, of course, the nature of the linear constraints attached to them. All problems are subject to the price feasibility constraint, i.e., all prices must be greater than or equal to zero.

Table 5. The matrix D[a]

i	d_{i1}	d_{i2}	d_{i3}	d_{i4}	d_{i5}	d_{i6}
1	-15.99	1.062	.11506	.05145	.11417	
2	1.062	-4.896	.017093	.0082265	.028025	
3	.11506	.017093	-19.27	.02955	.09073	
4	.05145	.0082265	.02955	-2.808	.044095	
5	.11417	.028025	.09073	.044095	-70.19	
6						-5.073

[a]In explanation, if, for example, i was set equal to 3, we would be referring to the third row of elements in the matrix D: .11506, .017093, etc.

B. Constraints

1. Total quantity

When, in a problem, the amount of milk used in the production of each of the products is to be allowed to vary within some range, but the total amount of milk available is to be fixed, then it is necessary to add a total quantity constraint. The total quantity of milk available in the t-th year is defined to be

$$TQ_t^F = Q_{1t}^F + Q_{2t}^F + Q_{3t}^F + Q_{4t}^F + Q_{5t}^F + Q_{6t}^F + X_{2t}^F + X_{3t}^F + X_{5t}^F.$$

Therefore, the total quantity constraint for the t-th year is found simply by adding together the total farm-level demand equations for the t-th year, setting the sum equal to the total amount of available milk for that year, and then subtracting the constant term on the left-hand side of the equation from the right-hand side of the equation. It should be noted that since the right-hand side term in the total quantity constraint was negative for all years, the constraint multiplied through by a minus one to satisfy the algorithm. The final form of this constraint is

$$L_t = 14.71144\ p_1^F + 3.839311\ p_2^F + 18.92193\ p_3^F + 2.65481\ p_4^F + 69.90571\ p_5^F + 5.073\ p_6^F$$

where L = 412.73, 335.81, 326.50, and 312.03 for 1951, 55, 60, and 64.

2. Welfare

a. Minimum consumption level   This constraint was designed to put lower limits on the domestically available quantities of the six dairy products. This was deemed desirable as a welfare constraint in view of the protests from consumers that would undoubtedly result because of the threat that large cutbacks in quantities would pose for nutritional and standard of living levels.

The minimum consumption levels for this constraint are based on a major food consumption survey, taken in 1955, of households in the United States (42). This survey gives household consumption levels for most of the foods with a breakdown into income groups. The minimum household consumption levels were arbitrarily selected to be those levels consumed by households in the $2000 to $3000 income bracket. In this particular study, this group of households appeared to have adequate nutrition and standard of living with respect to consumption of dairy products, according to Tobey (39). Per capita consumption figures were found by dividing household consumption figures by the average family size given in the survey for this group.

Total minimum consumption levels for the t-th year could have been found for all dairy products by multiplying per capita figures by total population in the t-th year. However, this method would fail to take account of changes in tastes for dairy products. This problem is resolved as shown in the following product by product derivation of minimum consumption levels.

Fluid milk and cream: Treating 1955 as the base year (as will be the case for all six products), the decrease in taste due to emphasis on low-fat diets was taken account of by multiplying the 1955 total minimum consumption level figure by the ratio of production in the t-th year to production in the year 1955.

Evaporated and condensed milk: The total minimum consumption level figure was adjusted to account for decreasing taste by using the production-ratio method described above.

Cheese: The total minimum consumption level was adjusted to take account of the increasing taste for cheese by using the production-ratio method.

Ice cream: The total minimum consumption level was adjusted to take account of the increasing taste for ice cream by using the production-ratio method.

Butter: The total minimum consumption level was adjusted to take account of the increasing substitution of margarine for butter by using the production-ratio method.

Other uses: The minimum consumption level used in the constraint represents a downward adjustment of actual consumption. This adjustment is partially based on the total minimum consumption level for nonfat dry milk and partially on an arbitrary figure assigned for the remaining uses of milk in this category. The minimum consumption level quantities are given in Table 7. The minimum consumption level constraint for each of the products in the t-th year is now found by setting each farm-level domestic use demand equation greater than or equal to its respective derived minimum consumption level for the t-th year.

b. Concentration inhibition   In light of minimum consumption levels, this constraint is designed to protect the processors and manufacturers of dairy products from too large a cutback in volume and to take into account the increased concentration that may result for a product industry when volume is decreased. The basis for determining this constraint is the per cent of the actual consumption level that the difference between the minimum consumption level and the actual consumption level is. These percentages are given in Table 6. The percentages for other uses were not computed because of the relative unimportance of the products which make up these other uses.

From Table 6 it appears that, with the exception of cheese, the constraint could be set up by utilizing the minimum consumption constraint levels given in Table 7. This was the chosen procedure. The constraint quantity for cheese, on the other hand, was arbitrarily determined by taking the yearly averages of the percentages given in Table 6 for evaporated and condensed milk, cheese, and

Table 6. Percent that actual consumption level minus minimum consumption level is of actual consumption level

Year	Fluid	Evaporated	Cheese	Ice Cream	Butter
51	5.7	12.9	67.4	12.7	12.4
55	5.7	12.9	67.3	12.6	12.4
60	5.7	13.0	67.3	13.1	12.4
64	5.7	12.9	67.5	13.0	12.4

Table 7. Actual and minimum consumption levels in million hundredweight

Year	Type	Fluid	Evaporated	Cheese	Ice Cream	Butter	Other
1951	Actual	541.00	64.01	116.70	70.01	303.63	51.32
	Minimum	510.19	55.75	38.10	61.33	265.02	30.23
1955	Actual	575.00	58.27	141.10	81.71	324.52	48.51
	Minimum	542.25	50.74	46.10	71.57	283.58	29.90
1960	Actual	585.00	53.20	152.80	94.53	301.73	40.86
	Minimum	551.68	46.27	49.90	82.81	262.27	26.74
1964	Actual	592.00	47.83	177.00	102.44	307.64	42.08
	Minimum	558.28	41.64	57.50	89.74	267.76	25.52

butter, and multiplying one minus this average percentage times actual cheese consumption. The quantities for this "concentration inhibition constraint" are given in Table 8. The constraint for each of the products is found by setting each farm-level total demand equation greater than or equal to its respective concentration inhibition level.

c. Price constraint The price constraint used in this study is the same as the one described earlier in Chapter two in connection with the optimal use of milk in the Netherlands. Thus, the form of the constraint for the t-th year is

$$I_t = w_{1t}\frac{p^F_{1t} - \overline{p}^F_{1t}}{\overline{p}^F_{1t}} + w_{2t}\frac{p^F_{2t} - \overline{p}^F_{2t}}{\overline{p}^F_{2t}} + w_{3t}\frac{p^F_{3t} - \overline{p}^F_{3t}}{\overline{p}^F_{3t}}$$

$$+ w_{4t}\frac{p^F_{4t} - \overline{p}^F_{4t}}{\overline{p}^F_{4t}} + w_{5t}\frac{p^F_{5t} - \overline{p}^F_{5t}}{\overline{p}^F_{5t}}$$

$$+ w_{6t}\frac{p^F_{6t} - \overline{p}^F_{6t}}{\overline{p}^F_{6t}} .$$

$$= w'_{1t}\, p^F_{1t} + w'_{2t}\, p^F_{2t} + w'_{3t}\, p^F_{3t} + w'_{4t}\, p^F_{4t} + w'_{5t}\, p^F_{5t}$$

$$+ w'_{6t}\, p^F_{6t} - 10,$$

$\overline{p}^F_{it}$, i = 1, 2, 3, 4, 5, and 6, being the "norm" prices (the "norm" prices, as in the study by Louwes *et al*. (29), are taken to be the actual prices), I_t an index of disutility, and w_{it}, i=1, 2, 3, 4, 5, and 6, appropriate weights.

Table 8. Actual and concentration inhibition levels in million hundredweight

Year	Type	Fluid	Evaporated	Cheese	Ice Cream	Butter	Other
1951	Actual	541.00	64.01	116.70	70.01	303.63	51.32
	Minimum	510.19	55.75	101.53	61.33	265.02	30.23
1955	Actual	575.00	58.27	141.10	81.71	324.52	48.51
	Minimum	542.25	50.74	122.76	71.57	283.58	29.90
1960	Actual	585.00	53.20	152.80	94.53	301.73	40.86
	Minimum	551.68	46.27	132.94	82.81	262.27	26.74
1964	Actual	592.00	47.83	177.00	102.44	307.64	42.08
	Minimum	558.28	41.64	153.99	89.74	267.76	25.52

The downward adjustment of the budget share of butter was accomplished by halving butter's share, the reason again being that margarine can be readily substituted for butter. The weights, w_{it}, scaled such that they sum to 10 for each year, are given in Table 9, and the price constraint coefficients, w'_{it}, are given in 10.

C. Formulation of Problems

Different combinations of the above-described constraints were added to the objective function to form the following maximization problems, each of which was solved for the years 1951, 1955, 1960, and 1964. The addition of constraints for each problem is not cumulative: only the specific constraints mentioned in the problem description, and no others, were added.

Problem(1): No constraints were added before maximizing the value of the objective function. That is, the problem is to maximize cash receipts of producers when total quantity of milk and allocation of milk among the six products is allowed to vary.

Problem(2): The total quantity constraint is added, and so the problem is to maximize cash receipts when only the allocation of milk among the products is allowed to vary.

Problem(3): The minimum consumption level welfare constraints are added before maximizing cash receipts of

Table 9. Weights, for each year, used in the price constraint

Year	Fluid	Evaporated	Cheese	Ice Cream	Butter	Other
1951	6.581	.624	.924	.573	1.022	.276
1955	6.775	.485	.957	.584	.961	.238
1960	6.805	.431	1.017	.663	.885	.199
1964	6.632	.378	1.159	.759	.874	.198

Table 10. Price constraint coefficients, w'_{it} [a]

t	w'_{1t}	w'_{2t}	w'_{3t}	w'_{4t}	w'_{5t}	w'_{6t}
1951	1.157	.148	.248	.150	.325	.110
1955	1.355	.147	.322	.192	.384	.114
1960	1.400	.137	.354	.226	.363	.099
1964	1.445	.124	.415	.250	.377	.103

[a]For example, if i was equal to 1 and t was equal to 1964, then $w'_{it} = w'_{1,\ 1964} = 1.445$.

producers.

Problem(4): The minimum consumption level and total quantity constraints are added before maximization.

Problem(5): The "concentration inhibition" welfare constraints are added before maximization.

Problem(6): The "concentration inhibition" and total quantity constraints are added.

Problem(7): The price constraint with $I_t = 0$ is added.

Problem(8): The price constraint with $I_t = 0$ and the total quantity constraint are added.

Problem(9): The price constraint with $I_t = 1$ is added.

Problem(10): The price constraint with $I_t = 1$ and the total quantity constraint are added.

Problem(11): The price constraint with $I_t = 2$ is added.

Problem(12): The price constraint with $I_t = 2$ and the total quantity constraint are added.

Problem(13): The price constraint with $I_t = 3$ is added.

Problem(14): The price constraint with $I_t = 3$ and the total quantity constraint are added.

Problem(15): The price constraint with $I_t = 0.6$, which is the approximate average change in the consumer price index between the years studied, is added.

Problem(16): The price constraint with $I_t = 28$, which is the approximate average I_t needed to give results as obtained in problem 1, is added.

Still further indices are possible for the price constraint, but the six indices used in this study were thought to be the most meaningful. The next chapter gives a complete discussion of the results obtained when these problems were solved.

VI. RESULTS

A. Presentation

Solutions to the problems studied, i.e., farm prices, retail prices, farm unit quantities, and individual product and total cash receipts, are given in the Appendix for the years 1951, 55 and 60, whereas solutions for 1964 are given in Tables 11 through 14 inclusive. The retail prices were estimated using a modified version of equation 3.3,

$$p_i^R = \frac{1 - \overline{r}_i}{.9} p_i^F + \frac{\overline{\overline{k}}_i}{.9} ,$$

where $\overline{r}_i$, as given in Brandow (8), is assumed to be the same for the four years studied and $\overline{\overline{k}}_i$ is equal to k_i for the particular year of the study. In keeping with Chapter three, $\overline{\overline{k}}_i$ was computed as $F_i - .1\ p_i^R$, where data for F_i were computed by the United States Department of Agriculture (45). For comparison, the Appendix and Tables 11 through 14 inclusive contain actual market data for the years studied. The reason for including only 1964 solutions and data in this chapter is given in the following section.

B. Discussion

Because, in most cases, the solutions to the problems studied showed the same basic results for each year, it is appropriate to summarize only the 1964 solutions, and discuss any significant differences for other years. The

Table 11. Farm prices in dollars per hundredweight for 1964: actual and solutions

Type	Fluid	Evaporated	Cheese	Ice Cream	Butter	Other
Actual	4.59	3.04	2.79	3.04	2.32	1.93
Problem 1	21.54	10.95	5.99	20.24	3.41	5.11
Problem 2	16.18	5.55	0.68	14.97	0.0	0.0
Problem 3	6.99	6.64	6.05	7.65	2.91	5.11
Problem 4	6.95	6.57	3.76	7.59	1.13	2.83
Problem 5	6.57	5.44	3.65	7.61	2.90	5.11
Problem 6	6.88	5.49	3.65	7.59	1.23	2.93
Problem 7	4.71	2.72	1.99	3.90	2.40	1.47
Problem 8	4.66	2.75	2.07	3.85	2.51	1.56
Problem 9	5.24	2.97	2.12	4.41	2.43	1.59
Problem 10	5.26	2.96	2.07	4.53	2.35	1.54
Problem 11	5.77	3.23	2.24	4.93	2.46	1.70
Problem 12	5.88	3.16	2.06	5.12	2.19	1.51
Problem 13	6.30	3.49	2.37	5.44	2.49	1.82
Problem 14	6.51	3.34	2.04	5.64	2.04	1.47
Problem 15	5.03	2.87	2.07	4.21	2.42	1.54
Problem 16	19.51	9.96	5.50	18.27	3.29	4.67

Table 12. Retail prices in dollars per hundredweight for 1964: actual and solutions

Type	Fluid	Evaporated	Cheese	Ice Cream	Butter
Actual	10.09	7.11	4.79	9.92	3.26
Problem 1	28.90	15.45	8.04	25.97	3.97
Problem 2	22.95	9.67	2.41	20.91	0.70
Problem 3	12.75	10.83	8.10	13.88	3.49
Problem 4	12.70	10.76	5.68	13.83	1.78
Problem 5	12.28	9.55	5.56	13.85	3.48
Problem 6	12.63	9.60	5.56	13.83	1.88
Problem 7	10.22	6.64	3.80	10.28	3.00
Problem 8	10.16	6.67	3.88	10.24	3.11
Problem 9	10.81	6.91	3.94	10.77	3.03
Problem 10	10.83	6.90	3.88	10.89	2.96
Problem 11	11.39	7.19	4.06	11.27	3.06
Problem 12	11.52	7.11	3.87	11.46	2.80
Problem 13	11.98	7.46	4.20	11.76	3.09
Problem 14	12.22	7.30	3.85	11.95	2.66
Problem 15	10.57	6.80	3.88	10.58	3.02
Problem 16	26.65	14.39	7.52	24.08	3.86

Table 13. Farm unit quantities in million hundredweight for 1964: actual and solutions

Type	Fluid	Evap.[a]	Cheese	I.C.[b]	Butter	Other	X-E[c]	X-C[d]	X-B[e]	Total
Actual	592.00	47.83	177.00	102.44	307.64	42.08	3.27	-6.70	.42	1265.98
Problem 1	331.04	46.96	132.79	54.87	235.39	25.95	-15.71	-22.70	-0.89	787.70
Problem 2	409.43	54.23	207.87	69.15	469.21	51.87	-2.75	3.85	3.20	1266.06
Problem 3	558.31	41.64	130.78	89.73	267.81	25.95	-5.37	-23.00	-0.29	1085.56
Problem 4	558.24	41.63	163.33	89.74	390.35	37.51	-5.20	-11.55	1.84	1265.89
Problem 5	563.36	44.11	165.00	89.73	268.17	25.95	-2.49	-11.00	-0.28	1142.55
Problem 6	558.24	44.25	164.90	89.73	383.42	37.01	-2.61	-11.00	1.72	1265.66
Problem 7	589.68	48.75	188.44	100.00	302.09	44.41	4.04	-2.70	0.32	1275.03
Problem 8	590.53	48.63	187.30	100.15	294.50	43.96	3.97	-3.10	0.19	1266.08
Problem 9	581.52	48.72	186.63	98.59	300.11	43.80	3.44	-3.35	0.28	1259.74
Problem 10	581.18	48.76	187.34	98.25	305.63	44.06	3.47	-3.10	0.38	1265.97
Problem 11	573.38	48.66	184.96	97.16	298.14	43.25	2.82	-3.95	0.25	1244.67
Problem 12	571.49	48.94	187.52	96.61	316.76	44.21	2.99	-3.05	0.57	1266.04
Problem 13	565.23	48.60	183.15	95.75	296.16	42.64	2.19	-4.60	0.21	1229.33
Problem 14	561.62	49.18	187.85	95.16	327.20	44.41	2.55	-2.95	0.75	1265.77
Problem 15	584.75	48.73	187.33	99.15	300.77	44.06	3.68	-3.10	0.30	1265.67
Problem 16	362.24	47.17	139.61	60.31	243.31	28.18	-13.33	-20.25	-0.75	846.49

[a] Evaporated

[b] Ice Cream

[c] Net export of Evaporated

[d] Net export of Cheese

[e] Net export of Butter

Table 14. Cash receipts in millions of dollars for 1964: actual and solutions

Type	Fluid	Evap.[a]	Cheese	I.Cream[b]	Butter	Other	X-E[c]	X-C[d]	X-B[e]	Total
Actual	2717.28	145.40	493.83	311.42	713.72	81.21	9.94	-18.69	0.97	4455.08
Problem 1	7130.59	514.21	795.41	1110.57	802.68	132.60	-172.02	-135.97	-3.0	10175.02
Problem 2	6624.57	300.98	141.35	1035.18	0.0	0.0	-15.26	2.62	0.0	8089.42
Problem 3	3902.59	276.49	791.22	686.43	779.33	132.60	-35.66	-139.15	-0.84	6393.00
Problem 4	3879.77	273.51	614.12	681.13	441.09	106.15	-34.16	-43.43	2.08	5920.25
Problem 5	3701.27	239.96	602.25	682.85	777.69	132.60	-13.55	-40.15	-0.81	6082.11
Problem 6	3840.69	242.93	601.88	681.05	471.61	108.44	-14.33	-40.15	2.12	5894.22
Problem 7	2777.39	132.60	375.00	390.00	727.02	65.28	10.99	-5.37	0.77	4471.66
Problem 8	2751.87	133.73	387.71	385.58	739.19	68.50	10.92	-6.42	0.48	4471.55
Problem 9	3047.16	144.70	395.66	434.78	729.27	69.64	10.22	-7.10	0.68	4824.99
Problem 10	3057.01	144.33	387.79	445.07	718.23	67.85	10.27	-6.42	0.89	4825.02
Problem 11	3308.40	157.17	414.31	479.00	733.42	73.52	9.11	-8.85	0.61	5166.69
Problem 12	3360.36	154.65	386.29	494.64	693.70	66.76	9.45	-6.28	1.25	5160.80
Problem 13	3560.95	169.61	434.07	520.88	737.44	77.60	7.64	-10.90	0.52	5497.80
Problem 14	3656.14	164.26	383.21	536.70	667.49	65.28	8.52	-6.02	1.53	5477.11
Problem 15	2941.29	139.86	387.77	417.42	727.86	67.85	10.56	-6.42	0.73	4684.92
Problem 16	7067.30	469.81	767.85	1101.86	800.49	131.60	-132.77	-111.37	-2.4	10092.29

[a] Evaporated

[b] Ice Cream

[c] Net export of Evaporated

[d] Net export of Cheese

[e] Net export of Butter

problems will be grouped in logical subdivisions for discussion purposes: basic problems (Problems 1 and 2), minimum consumption level problems (Problems 3 and 4), "concentration inhibition" problems (Problems 5 and 6), and price constraint problems (Problems 7 through 16 inclusive). Again, keep in mind that all discussion refers to 1964 solutions unless otherwise stated.

1. Problems 1 and 2

The results of Problems 1 and 2 are extremely favorable for milk producers. For Problem 1 they show that by decreasing the total quantity of milk available by 38 percent to a figure of 787.7 million hundredweight and allocating the milk among the six products in a specified way, milk producers as a whole could have raised their total cash receipts from $4,455 million to $10,175 million (an increase of 103 percent) in 1964. This increase in total cash receipts would involve an increase in domestic cash receipts for every product (all product figures will be in milk equivalent). Fluid milk, evaporated milk, and ice cream would show the largest relative cash receipts increases and butter the smallest. In 1964, cash receipts for fluid, evaporated, and ice cream would have been increased 162, 248, and 257 percent respectively. Cash receipts from net exports would have declined for all exported products because of increased domestic prices: exports of evaporated milk and cheese, for instance, by 182 and 118 million hundredweight respectively in 1964. Concerning the prices received by farmers, the farm price for fluid milk would be $21.54 per hundredweight (an increase of 370 percent), and the farm price for ice cream would have increased 566 percent to a level of $20.24. The smallest percentage increase in a farm price for 1964 was that of butter (47 percent).

Although the producer side of Problem 1 is very bright, the consumer side is equally dim. For example, the above-mentioned quantity cut and prescribed allocation would have cut the amount of fluid milk from 592 to 331 million hundredweight in 1964, while at the same time increasing the retail fluid milk price to $28.90 per hundredweight. Decreases in the quantities of evaporated milk, cheese, ice cream, butter, and other uses would have been 2, 25, 47, 23, and 38 percent respectively in 1964.

Problem 2 shows that producers could have increased total cash receipts by 45 percent in 1964 while still having produced the same total quantity of milk. This increase could have been made possible by increasing the amounts of milk allocated to the production of evaporated milk, cheese, butter and other uses by 13, 18, 52, and 24 percent respectively, while decreasing the amounts of milk allocated to the production of fluid milk and ice cream by 45 and 32 percent respectively. This reallocation of quantity would cause large increases in the farm prices of fluid, evaporated, and ice cream and large decreases in the prices of the other three products. Notice that butter became an abundant product (prices equal to zero) here for 1955, 60, and 64, and other uses became abundant in 1964.

The interesting thing about Problem 2 is that it shows that excess milk in the market should be allocated to butter production first, then other uses, then cheese, in order to work toward maximizing cash receipts of producers.

2. Problems 3, 4, 5, and 6

a. Problems 3 and 4 Problems 1 and 2 called for changes in quantities and prices that would be highly unacceptable socially and politically. Problem 3, on the other hand, sets an acceptable minimum level for quantities, which in turn leads to prices that are much less undesirable from the consumer standpoint than those of Problems 1 and 2. In evidence of this, for fluid milk the 1964 farm price is \$6.99 per hundredweight in Problem 3 compared to \$21.54 per hundredweight in Problem 1, and for ice cream it is \$7.65 per hundredweight compared to \$20.24 per hundredweight. The price of \$6.99 for fluid milk at the farm level would mean a 26 percent increase in the retail price for 1964.

The results for Problem 3 show that, in every year, the quantities of fluid milk, evaporated milk, ice cream, butter and other uses are equal to their minimum consumption constraint levels, whereas the quantity of cheese in the solution is greater than its minimum level. It can be said that the constraint on cheese was trivial for this problem, and, therefore, did not affect the value of the objective function. The total quantity of milk available was cut by 6 percent in 1964 for Problem 3.

The additional milk made available in Problem 4 by adding the total quantity constraint caused a decrease of

$473 million in total cash receipts for 1964 when compared to Problem 3. The additional milk was allocated to the production of butter, cheese, and other uses, 68 percent of it being used for butter production.

b. Problems 5 and 6 Setting a lower limit on cheese production led to a decrease of $311 million in total cash receipts for Problem 5 compared to Problem 3 in 1964. Involved here were decreases in cash receipts for fluid milk, evaporated milk and cheese of 5, 13, and 24 percent respectively. The change in cash receipts for ice cream, butter, and other uses was negligible. Farm prices of fluid milk, evaporated milk, and cheese in Problem 5 were the only ones with significant differences from the prices of Problem 3, the cheese price having the largest difference (a 40 percent decrease for 1964). The total quantity result in Problem 5 represented a 5 percent increase in total milk available compared to Problem 3 in 1964, with ice cream, butter, and other uses quantities equal to constraint quantities, and fluid milk, evaporated milk, and cheese quantites just slightly above constraint quantities. The results of Problem 6 were very similar to those of Problem 4.

3. Problems 7 through 16 inclusive

Problem 7, as expected, did not show any significant increase in total cash receipts of dairy farmers. The increase in those receipts was only $17 million in 1964 (less than ½ percent). There were slight increases in the quantities of evaporated milk, cheese, butter, and other uses, and slight decreases in fluid milk and ice cream quantities. These quantity changes were accompanied by decreases in farm prices of evaporated milk, cheese, and other uses by 11, 29, and 24 percent respectively, and increases in the prices of fluid milk, ice cream, and butter by 3, 28, and 3 percent respectively. (The farm price of butter also increased slightly in 1955 and 1960, but decreased in 1951). The total quantity of milk available was actually increased by 9 million hundredweight in this problem for 1964. The addition of the total quantity constraint to Problem 7 did not produce any significant changes in the solutions.

Setting the price constraint index at one as in Problem 9 (recall that this represents a 10 percent increase in the relative farm price level) results in an increase of 8 percent in total cash receipts for 1964. Cash receipts for cheese exports increased while cash receipts for exports of evaporated milk and butter stayed about the same. In 1964, cash receipts for fluid milk and ice cream increased 12 and 40 percent respectively, cash receipts for evaporated milk and butter changed only slightly, and cash receipts for cheese and other uses decreased 25 and 16 percent respectively. Also in 1964, significant changes in farm prices occurred for fluid milk (15 percent increase), ice cream (38 percent increase), and butter (39 percent decrease). Retail price changes in 1964 for these same products were a 9 percent increase, a 6 percent increase, and a 27 percent decrease respectively. Total quantity of milk available was about the same as for actual data. Again, as in Problem 7 and 8, results for Problem 10 were very similar to those of Problem 9. However, a trend, to be discussed below, has begun to develop.

Problems 11, 12, 13 and 14 showed a continuation of the trends shown in Problems 7, 8, 9 and 10. In Problem 13 there is an increase in total cash receipts from $4455 million to $5498 million or a 23 percent increase. Cash receipts of fluid milk and ice cream, as in Problem 11, increased substantially for Problem 13 in 1964, while those of cheese decreased. Problem 13 also shows the cumulative effect of having continually raised the price index, eventually reaching a value of three. All farm prices smoothly rise as the index is increased without the total quantity constraint, but the farm prices for cheese and other uses are still 12 and 6 percent less than the actual farm price respectively for 1964. That is, as the price constraint index is raised, the solution prices asymptotically approach those of Problem 1. In Problem 13, farm prices of fluid, evaporated, ice cream, and butter have increased 37, 15, 79, and 7 percent respectively over actual prices. Total quantity decreased only 3 percent in 1964, with a 5 percent decrease in the quantity of fluid milk, a 6 percent decrease in the quantity of ice cream, a 4 percent decrease in butter and a negligible change for evaporated milk and other uses. Net export quantities were about the same as actual quantities.

Problems 8, 10, 12, and 14 show that when the total quantity constraint is added to the price constraint problem, this forces an increase (decrease) in the prices (quantities) of fluid, evaporated, and ice cream, and a decrease (increase) in the prices (quantities) of cheese, butter, and other uses. In addition, as the price index is increased in these problems, prices of fluid, evaporated, and ice cream are increased and those of cheese, butter and other uses decreased. It is apparent that, as the price constraint index is raised when the total quantity constraint is present, the solution prices asymptotically approach those of Problem 2.

Problems 15 and 16 are special interest cases for the price constraint when the total quantity constraint is not added. The first of these problems shows that, had a weighted average of the relative farm prices increased at the same rate as did the weighted average of all consumer prices, then cash receipts of milk producers would have increased an average of 5 percent between each year studied, or a total of 15 percent between the years 1951 and 1964. Problem 16 was designed to approximate, using the price constraint, the solutions to Problem 1. Problem 1, for the years 1951, 55, 60, and 64, implies price constraint indexes of 21.1, 27.5, 29.7, and 31.9 respectively.

4. Requests for price increases

As was cited earlier, the National Farmers' Organization is attempting, by their holding action, to obtain a contracted two cents per quart ($.93 per cwt.) increase in the farm price of fluid milk. At the same time, the National Milk Producers Federation is lobbying congress to establish a one cent per quart ($.47 per cwt.) raise in the farm price of fluid milk. The increase which the N.F.O. is bargaining for, if obtained, would bring about an increase in total cash receipts of $551 million (a 12 percent increase), which involves a 20 percent increase in cash receipts for fluid milk. Also, the 2 cents per quart would be a 20 percent increase in the farm price and would mean a 9 percent increase in the retail price.

The 1964 solutions to Problem 15, if instrumented, would give an increase in total cash receipts approximately equal to the target level of the National Milk Producer's Federation. The increase in total cash receipts which would result from the fulfillment of the N.F.O.'s plan could be achieved by adopting, as instrument variables, quantities about midway between the solution quantities of Problems 9 and 11, i.e., solutions to a problem with price constraint index equal to 1.5.

C. Sensitivity

In the quadratic programming problem

$$\text{maximize } z = cx + x'Dx \qquad \text{subject to } Ax = b \text{ and } x \geq 0,$$

the elements of c, D, A, and b are usually not exactly known, as was pointed out to be the case for the problems in this study. Therefore, it is helpful to determine how the solution changes with a change in the values of these elements. Such an investigation is termed a sensitivity analysis or a check on the robustness of the solutions.

Detailed expositions of the theory of sensitivity analysis, such as those of Boot (6 and/or 7) and Theil (38), are available for the interested reader. However, given that D is negative semi-definite, the more general results of this theory are

$$\frac{\partial z}{\partial c_i} \geq 0, \ \frac{\partial z}{\partial d_i} \leq 0, \ \frac{\partial z}{\partial a_{ij}} \leq 0, \text{ and } \frac{\partial z}{\partial b_i} \geq 0.$$

The last two results apply only when given nonnegativity conditions and only inequality constraints. Note the offsetting effects that certain changes in the elements will have. For instance, changes of the elements of c and D in the same direction will have offsetting effects on the solution. One thing that becomes apparent in performing a sensitivity analysis is that you must be careful not to change the negative definiteness of D when altering its elements.

The scope of the sensitivity analysis performed in this study was necessarily limited. Nevertheless, the analysis was large enough to convey meaningful conclusions. It first involved a study of Problem 2 in hopes of finding solution values which were free of abundancy. Next, the minimum consumption levels of Problems 3 and 4 were systematically varied for 1960 and 1964. Since Problems 5 and 6 involved only a difference in the level of cheese production when compared to Problems 3 and 4, Problems 5 and 6 were not analyzed. This is obviously of little consequence to the sensitivity analysis. Lastly, using the Monte Carlo approach, a sensitivity analysis of significant proportions was carried out on Problems 1 and 9 for 1960 and 1964. Problem 1 was chosen for analysis because it is one of the two most basic problems studied, and Problem 9 was chosen because it is fairly representative of the many problems solved with the price constraint added. A Monte Carlo approach, when done on this scale, is likely to show the influence on the solutions of the fact that the elasticities were uncertain.

1. Analysis of Problem 2

The analysis of Problem 2 consisted of experimenting with different conbinations of farm-demand slope coefficients for butter and other uses for the years 1955, 60, and 64. The slope coefficient for butter was allowed to vary from -100.0 to -40.0 in steps of 20.0, and that of other uses was allowed to vary from -6.5 to -3.5 in steps of 1.0. All 16 possible combinations of these ranges and steps were introduced into Problem 2, and the problem was then solved. It was found that most alternative farm-demand slope coefficient combinations attempted (note that because of the way in which the model is set up that a change in the farm-demand equation slope coefficient changes the constant term of the equation) left (see page 77) butter and other uses milk abundant for each year and resulted in cheese becoming abundant. All of these alternative combinations left at least cheese and butter abundant and gave values of the objective function from 6 to 12 per cent lower. In addition, attempts at fixing the prices of abundant products at some value greater than zero were futile, because fixing the price of an abundant product at an arbitrarily reasonable level caused previously non-abundant products to become abundant. Such price fixing can be done only at cost to the value of the objective function. The reason for abundancy in 1955, 60, and 64, whereas no abundancy occurred in 1951, is, of course, partially due to the fact that the total quantity of milk available for the year 1951 was about 90 million cwt. less than the average available for the other three years.

2. Analysis of Problems 3 and 4

Since the final form of the minimum consumption level constraint is a set of less than or equal to inequality equations, the previously cited theorem,

$$\frac{\partial z}{\partial b_i} \geq 0,$$

is applicable. Lowering the minimum consumption levels for the products corresponds to increasing the right hand sides of the inequality constraints, and vice versa, and, lowering the minimum consumption levels increases the value of the objective function, and vice versa.

Four different variations of the levels were used in the analysis: a 5 percent increase, a 5 percent decrease, a 15 percent decrease, and a 25 percent decrease. For a 5 percent increase in levels there was a 16 percent average decrease in cash receipts for Problem 3 and a 15 percent decrease for Problem 4. Price results were basically the same as in the original problems except for the appropriate downward adjustments. That is, prices of all products in Problem 3 were greater than their actual prices, and prices in Problem 4 were greater except for butter, whose price was less than the actual price. For a 5 percent decrease in minimum consumption levels, there was, of course, an increase in cash receipts for Problems 3 and 4. As the minimum consumption levels were further decreased, the solutions approached those that were arrived at in Problems 1 and 2. This was certainly to be expected. With a 25 percent decrease in minimum levels, cash receipts rose 33 percent and 27 percent respectively for Problems 3 and 4, and butter became abundant for both years in Problem 4. In summary, cash receipts for Problems 3 and 4 are fairly sensitive to variations in the minimum consumption levels.

3. Monte Carlo approach analysis

The Monte Carlo approach used on Problems 1 and 9 consisted of selecting at random 20 sets of slope coefficients, and then solving the problems with these alternative models for the last two years of the study. The slope coefficients derived from direct price elasticities were selected from a range of coefficients based on results from all available elasticity studies, except for the coefficients for other uses of milk and exports of milk products. The ranges for these were set up by allowing a 30 percent deviation from the slopes used previously in this study. Table 15 gives the ranges arrived at for each product. Slopes used in the original problems are also given.

Table 15. Alternative slope ranges

Product	Low	High	Original
Fluid	-22.0	-10.0	-15.99
Evaporated	-6.7	-1.7	-2.496
Cheese	-18.0	-10.0	-14.27
Ice cream	-4.0	-2.0	-2.808
Butter	-100.0	-40.0	-68.99
Other	-6.5	-3.5	-5.073
X-Evaporated	-2.1	-1.7	-2.4
X-Cheese	-6.5	-3.5	-5.0
X-Butter	-1.6	-.8	-1.2

Since the values of slope coefficients corresponding to cross-elasticities are also uncertain, variations in these were added to the random selection process. These variations were made up of a 20 percent decrease in all cross-elasticity slope coefficients, a 10 percent decrease in all these slope coefficients, no change in them, and 10 and 20 percent increases for all cross-elasticity slope coefficients. The twenty sets of randomly selected slope coefficients are shown in Table 16. The D matrix resulting from each of these cases was checked for negative definiteness.

The ranges (R's) and means ($\overline{X}$'s), along with standard deviations (s's) and t-values (which are defined, for either Problem 1 or Problem 9 as the case may be, as (note that $\frac{s}{\sqrt{n}} = \sigma_{\overline{x}_{it}}$)

$$\frac{\overline{x}_{it} - p^F_{it}}{\sigma_{\overline{x}_{it}}},$$

where the p^F_{it} (i = 1, 2, 3, 4, 5, and 6; t = 1960 and 1964) are the original solution prices, $\overline{x}_{it}$ is the mean of the i-th prices generated in the Monte Carlo approach for the t-th year, and $\sigma_{\overline{x}_{it}}$ is the appropriate standard deviation of $\overline{x}_{it}$) for the price and total cash receipts results of Problems 1 and 9 in the Monte Carlo approach are given in Table 19. Tables 17 and 18 give, for each of Problems 1 and 9 respectively, price and total cash receipts solutions for each of the twenty alternative problems which resulted from the twenty sets of randomly selected slope coefficients.

Table 16. Set of twenty randomly selected models

Set	Fluid	Evap.[a]	Cheese	I.C.[b]	Butter	Other	X-Evap.[c]	X-Cheese[d]	X-Butter[e]	Percent Change in Cross Slopes
1	-14	-1.7	-12	-2.5	-60	-5.0	-1.8	-4.5	-1.6	+20
2	-20	-3.7	-18	-4.0	-70	-4.5	-1.9	-3.5	-1.0	-10
3	-22	-5.7	-14	-2.5	-90	-5.0	-2.1	-5.0	-.8	-10
4	-22	-3.7	-10	-3.5	-70	-6.0	-1.9	-5.5	-1.2	+10
5	-12	-4.7	-10	-2.0	-60	-4.5	-1.7	-6.5	-.8	+10
6	-22	-3.7	-16	-2.5	-70	-5.0	-1.9	-6.0	-1.6	+20
7	-18	-3.7	-14	-3.0	-70	-3.5	-2.1	-5.0	-1.2	-20
8	-12	-6.7	-14	-2.5	-90	-6.0	-1.7	-3.5	-1.0	0
9	-22	-3.7	-14	-2.0	-50	-5.0	-1.9	-5.5	-1.0	-20
10	-18	-4.7	-18	-3.0	-40	-5.5	-1.8	-5.0	-1.4	+10
11	-16	-1.7	-18	-3.5	-70	-6.0	-1.9	-4.5	-1.2	0
12	-10	-1.7	-12	-2.0	-70	-4.5	-2.1	-3.5	-1.4	+20
13	-12	-6.7	-10	-4.0	-70	-6.0	-2.1	-5.0	-.8	+10
14	-18	-6.7	-12	-3.5	-50	-6.0	-1.7	-6.5	-1.2	0
15	-20	-3.7	-14	-2.5	-60	-5.0	-1.7	-4.5	-.8	0
16	-10	-5.7	-16	-4.0	-80	-5.5	-2.1	-5.0	-1.0	-20
17	-18	-4.7	-16	-2.0	-90	-3.5	-1.9	-6.0	-1.4	+10
18	-18	-3.7	-18	-3.5	-40	-5.5	-2.0	-4.5	-1.2	-10
19	-22	-4.7	-12	-3.0	-70	-4.0	-2.0	-6.5	-1.0	0
20	-10	-4.7	-16	-2.5	-50	-4.5	-2.1	-6.5	-1.2	0

[a] Evaporated

[b] Ice cream

[c] Net export of evaporated

[d] Net export of cheese

[e] Net export of butter

Table 17. Price (in dollars per hundredweight) and total cash receipts (in millions of dollars) solutions for each of the twenty alternative problems which resulted from the twenty sets of randomly selected slope coefficients for Problem 1

Set	Year	Fluid	Evaporated	Cheese	Ice Cream	Butter	Other	Total Cash Receipts
1	1960	24.97	18.00	6.19	21.13	3.76	5.09	11296.27
	1964	25.03	17.18	6.84	22.78	3.75	5.17	11590.10
2	1960	17.48	9.25	4.98	13.53	3.39	5.55	8430.36
	1964	17.49	8.69	5.47	14.57	3.37	5.64	8623.74
3	1960	16.02	6.91	5.44	20.72	2.91	5.09	8271.82
	1964	16.04	6.50	6.00	22.37	2.89	5.17	8531.45
4	1960	16.22	9.57	6.38	15.30	3.39	4.42	8236.04
	1964	16.22	9.02	7.07	16.50	3.37	4.47	8465.20
5	1960	27.88	10.72	6.20	26.02	3.79	5.55	12180.25
	1964	28.01	10.25	6.84	28.07	3.78	5.64	12578.99
6	1960	16.29	9.86	4.94	20.86	3.38	5.10	8442.99
	1964	16.29	9.31	5.41	22.50	3.36	5.17	8681.66
7	1960	19.09	8.95	5.45	17.54	3.38	6.84	9126.30
	1964	19.14	8.42	6.00	18.91	3.36	6.98	9380.10
8	1960	27.52	8.16	5.88	21.01	2.93	4.42	11644.57
	1964	27.67	7.79	6.49	22.66	2.90	4.47	11995.88
9	1960	16.07	8.77	5.34	25.52	4.24	5.10	8695.77
	1964	16.08	8.21	5.88	27.56	4.23	5.17	8995.21

Table 17. (continued)

Set	Year	Fluid	Evaporated	Cheese	Ice Cream	Butter	Other	Total Cash Receipts
10	1960	19.25	9.02	4.80	17.67	4.96	4.72	9294.91
	1964	19.30	8.54	5.25	19.06	4.97	4.79	9544.12
11	1960	21.70	15.22	4.86	15.35	3.40	4.42	9862.80
	1964	21.73	4.39	5.33	16.54	3.37	4.47	10064.60
12	1960	34.25	19.83	6.60	26.35	3.44	5.55	14286.47
	1964	34.40	19.10	7.28	23.40	3.42	5.64	14686.96
13	1960	27.57	8.17	6.63	13.73	3.42	4.41	11413.64
	1964	27.71	7.82	7.35	14.79	3.40	4.47	11729.66
14	1960	19.08	7.09	5.58	15.31	4.24	4.42	9033.48
	1964	19.13	6.71	6.16	16.51	4.23	4.47	9286.77
15	1960	17.58	9.87	5.58	20.82	3.76	5.09	8890.02
	1964	17.60	9.30	6.15	22.46	3.75	5.17	9158.17
16	1960	32.36	8.50	5.12	13.65	3.13	4.72	12685.57
	1964	32.56	8.10	5.61	14.69	3.11	4.79	12994.19
17	1960	19.26	8.91	4.95	25.73	2.92	6.85	9488.63
	1964	19.31	8.44	5.43	27.78	2.89	6.98	9793.24
18	1960	19.17	9.41	4.84	15.28	4.96	4.72	9170.27
	1964	19.20	8.86	5.30	16.47	4.96	4.79	9398.17
19	1960	16.10	8.02	5.57	17.56	3.39	6.12	8245.32
	1964	16.11	7.55	6.14	18.94	3.37	6.22	8482.91
20	1960	32.80	10.54	4.93	21.24	4.27	5.55	13350.54
	1964	32.98	10.10	5.39	22.78	4.27	5.64	13718.21

Table 18. Price (in dollars per hundredweight) and total cash receipts (in millions of dollars) solutions for each of the twenty alternative problems which resulted from the twenty sets of randomly selected slope coefficients for Problem 9

Set	Year	Fluid	Evaporated	Cheese	Ice Cream	Butter	Other	Total Cash Receipts
1	1960	5.49	3.76	2.10	4.20	2.63	1.52	4929.83
	1964	5.19	3.53	2.11	4.26	2.58	1.49	4833.95
2	1960	5.56	3.14	2.18	4.00	2.51	1.91	4901.56
	1964	5.25	2.92	2.20	4.07	2.46	1.86	4807.05
3	1960	5.37	2.71	2.31	5.68	2.24	1.86	4909.15
	1964	5.02	2.53	2.35	5.79	2.19	1.81	4813.30
4	1960	5.40	3.27	2.51	4.48	2.52	1.71	4891.08
	1964	5.08	3.03	2.55	4.53	2.46	1.65	4798.60
5	1960	5.56	2.67	2.03	4.55	2.63	1.52	4940.26
	1964	5.23	2.53	2.04	4.60	2.58	1.49	4844.37
6	1960	5.31	3.27	2.17	5.55	2.51	1.82	4909.27
	1964	4.99	3.03	2.18	5.59	2.45	1.77	4814.94
7	1960	5.51	2.91	2.18	4.46	2.48	2.02	4910.55
	1964	5.16	2.73	2.22	4.60	2.43	2.01	4815.67
8	1960	5.76	2.43	2.04	4.12	2.17	1.43	4948.02
	1964	5.45	2.31	2.05	4.18	2.12	1.39	4846.80
9	1960	5.10	2.96	2.21	6.23	3.00	1.76	4929.88
	1964	4.77	2.74	2.22	6.23	2.94	1.71	4839.44
10	1960	5.37	2.85	2.03	4.35	3.38	1.62	4955.68
	1964	5.05	2.66	2.04	4.40	3.33	1.57	4865.05

Table 18. (continued)

Set	Year	Fluid	Evaporated	Cheese	Ice Cream	Butter	Other	Total Cash Receipts
11	1960	5.62	3.78	2.01	3.80	2.47	1.54	4924.64
	1964	5.32	3.54	2.02	3.88	2.42	1.50	4827.07
12	1960	5.65	3.38	1.97	4.06	2.40	1.42	4943.95
	1964	5.34	3.20	1.98	4.12	2.35	1.40	4845.05
13	1960	5.83	2.45	2.15	3.16	2.44	1.43	4934.25
	1964	5.51	2.32	2.17	3.22	2.39	1.40	4836.99
14	1960	5.50	2.58	2.21	4.06	2.98	1.60	4925.77
	1964	5.18	2.42	2.23	4.11	2.93	1.55	4834.37
15	1960	5.31	3.13	2.24	5.20	2.71	1.75	4913.09
	1964	4.99	2.90	2.25	5.24	2.66	1.70	4820.46
16	1960	5.98	2.37	1.88	2.94	2.27	1.41	4957.73
	1964	5.67	2.26	1.88	3.01	2.22	1.38	4856.63
17	1960	5.41	2.85	2.07	5.81	2.20	1.98	4932.08
	1964	5.08	2.66	2.07	5.86	2.15	1.93	4833.32
18	1960	5.41	2.94	2.05	3.96	3.39	1.63	4952.15
	1964	5.10	2.75	2.06	4.03	3.35	1.58	4861.61
19	1960	5.37	2.94	2.34	4.97	2.52	2.06	4898.02
	1964	5.05	2.73	2.36	5.01	2.47	2.00	4805.19
20	1960	5.71	2.52	1.82	3.64	2.86	1.44	4966.28
	1964	5.38	2.41	1.84	3.81	2.83	1.44	4870.21

Table 19. Sensitivity analysis price and total cash receipts results, in dollars per hundredweight, for Monte Carlo approach

Statistic	Fluid	Evaporated	Cheese	Ice Cream	Butter	Other	Total Cash Receipts
Problem 1							
1960							
R	16.02-34.25	6.92-19.83	4.80-6.64	13.53-26.35	2.92-4.96	4.42-6.85	8236.04-14286.47
$\bar{x}$	22.03	10.24	5.51	19.21	3.65	5.19	10102.28
s	6.04	3.35	0.60	4.24	0.59	0.73	1842.62
t	0.37	-1.79	0.56	0.47	1.70	0.92	0.41
1964							
R	16.04-34.40	6.50-19.10	5.25-7.35	14.57-28.40	2.89-4.97	4.47-6.98	8465.20-14686.96
$\bar{x}$	22.10	9.71	6.07	20.72	3.64	5.27	10382.94
s	6.11	3.24	0.68	4.58	0.60	0.75	1890.79
t	0.41	-1.71	0.53	0.47	1.70	0.93	0.49
Problem 9							
1960							
R	5.10-5.99	2.38-3.76	1.82-2.51	2.94-6.23	2.17-3.39	1.41-2.06	4891.08-4966.28
$\bar{x}$	5.51	2.90	2.12	4.46	2.62	1.67	4928.65
s	0.21	0.35	0.16	0.85	0.34	0.21	21.47
t	-0.88	-3.63	0.71	0.59	1.79	0.92	1.54
1964							
R	4.77-5.68	2.27-3.53	1.84-2.55	3.01-6.23	2.12-3.35	1.38-2.01	4798.60-4870.21
$\bar{x}$	5.19	2.71	2.14	4.53	2.57	1.63	4833.50
s	0.21	0.31	0.16	0.84	0.34	0.20	20.24
t	-1.04	-3.67	0.59	0.63	1.77	0.95	2.04

The t-values show that, out of 24 mean prices in the Monte Carlo approach sensitivity analysis, none of them differ greatly from the original solution prices when the differences are considered in relation to the standard deviations of these mean prices. It is certainly possible to conclude that my results are at least qualitatively correct, and perhaps one might even say that they are approximately quantitatively so. In essence, the solutions are robust with respect to the uncertainty of the values of the elasticities.

D. Solution Accuracy

For all problems solved in this study a computation of row errors was made. That is, the computed values for the row restraints were compared with the original restraint values. This check showed that accuracy was good to the second or third decimal place for all problems. However, had accuracy for a particular problem been at an undesirable level, a forced inversion could have been executed. This forced inversion reduces inaccuracies associated with large problems and many iterations. On the contrary, the problems in this study were small and all required less than ten iterations.

VII. SUMMARY

This study found that it was possible for dairy farmers as a whole, whose economic position was outlined in Chapter one, to more than double their total cash receipts by cutting total milk available by more than a third. This increase in cash receipts would involve an increase in all domestic cash receipts, but a decrease in net exports cash receipts due to the increase in domestic prices. Most notable of these price increases is that of fluid milk, the price of which quadrupled.

Even while given the same total quantity of milk, total cash receipts of producers could be increased 45 percent. The reason that such a large increase can be obtained, and that optimal allocations differ so much from actual allocations, can be partially explained by the principles of price discrimination. Price discrimination theory says in general that if the elasticities of demand in the markets (in this case the six products of milk) differ, the higher price(s) will be charged in the less elastic market(s) (in this case the fluid milk market is less elastic than all of the other product markets), assuming separation of these markets. And, according to price discrimination theory, to get the greatest total revenue from any given total volume of sales where there are two or more separated markets, marginal revenue in each of the separated markets must be equated (which undoubtedly was not the case for actual market results cited in this study). That is, if the marginal revenue in one market is greater than that in the other, it will pay to shift sales (by raising price) out of the market in which the marginal revenue is low into the market in which it is high. However, it should be noted that since handlers are normally of the multimarket type (for example, a handler may manufacture cheese, ice cream, and butter in addition to processing fluid milk and cream), price discrimination theory cannot be rigorously applied in this study because we do not have separation of markets in the strict sense. In addition, price discrimination theory can also be applied, again not in a strict sense, when total volume of sales is not constrained. The results, although similar, are more complicated (26).

A further study result was that excess milk in the market should be allocated to butter production first, then other uses, then cheese, in order to move toward the maximization of cash receipts of milk producers.

The addition of constraints to Problems 1 and 2, both minimum product level and price constraints, revealed, among other things, that adding minimum product constraint levels lowers total cash receipts from 2 to 4 billion dollars, and that the solutions to a problem with price constraint index equal to about 1.5 would satisfy the desires of the National Farmers Organization. In addition, it should be noted that as the price constraint index is raised from zero, the solutions to the quadratic programming problem approach those of Problem 1 if the total quantity constraint is not added along with the price constraint, and the solutions approach those of Problem 2 if the total quantity constraint is added.

An extensive sensitivity analysis revealed some very interesting results. First of all, it was not possible, using excepted product elasticity ranges, to free Problem 2 of abundancy. Secondly, cash receipts for Problems 3 and 4 were fairly sensitive to variations in the minimum consumption levels. And lastly, and most interesting of all, a Monte Carlo sensitivity analysis of sizeable proportion showed that the solutions were robust with respect to the uncertainty of the values of the elasticities.

VIII. RECOMMENDATIONS AND CONCLUSIONS

If something is going to be done to enhance the position of the American dairy farmer, the best position to take might be to use the quantities found in Problem 15 as instrument variables at first, and eventually to work toward those product allocations of the total quantity of milk available which would increase cash receipts comparable to what the N.F.O. desires. Such a position may be desirable in view of the threat that exists to the United States' milk supply should no action be taken.

In searching for a means to bring about the desired levels of the instrument variables, let us first consider the bargaining power of the N.F.O. as a prime example of producer association bargaining power. Although the N.F.O. has had isolated success in its milk holding action, it appears highly unlikely that it will be entirely successful unless it can restrict the total supply of milk. The reason is that the higher prices which they are asking will mean less demand for milk but more milk produced: the law of demand will provide a strong test for the unity of the organization. But, as Ladd (25, p. 136) points out, "It is difficult, and probably impossible to form a large organization possessing sufficient unity to be able to survive the test." Indeed, besides the program recommended here requiring some restriction on the total supply of milk as milk prices begin to increase, it also requires a reallocation of the available quantity of milk among the production of the different products. The bargaining power needed for such regulation in addition to control of the milk supply would require even stronger unity in the controlling organization. Manufacturers of the various products would not agree easily to such outside control of their production, and so the organization would have to first prove that it could impose an economic loss on manufacturers if they did not cooperate or a gain to them if they did cooperate.

What is needed to get around the problem of unity is some type of governmental assistance, such as favorable legislation or administration. The existing federal milk marketing orders, by an act of congress, might be used to achieve the desired levels of production in each product, and also to control the response of supply to the higher cash receipts. This interproduct or interuse regulation would be a totally new concept in market orders, but maybe a very successful one. Interproduct regulation could be accomplished by dividing the use of milk into six classes (the six products), and then establishing a price for each class of milk such that the desired cash receipts would be achieved. In this instance prices are, of course, the instrument variables. The advantage of a marketing order is that a bargaining association, capable of imposing large economic losses on handlers, is not needed. Handlers could be expected to cooperate because a marketing order has the force of law and violators can be prosecuted.

The above plan would no doubt meet strong opposition by handlers, and would have to be very carefully studied before being put into operation. However, if the recent dissatisfaction of dairy farmers continues, such a marketing plan may be necessary to guarantee an adequate supply of milk in the future.

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XI. APPENDIX

Table 20. Farm prices in dollars per hundredweight for 1951, 55, and 60: actual and solutions

Year	Fluid	Evaporated	Cheese	Ice Cream	Butter	Other
			Actual			
1951	5.69	4.23	3.72	3.83	3.14	2.51
1955	5.00	3.30	2.97	3.04	2.50	2.08
1960	4.86	3.15	2.87	2.93	2.44	2.02
			Problem 1			
1951	20.56	13.01	5.03	14.83	3.78	6.31
1955	21.26	12.13	5.20	16.54	3.62	5.82
1960	21.54	11.58	5.44	18.77	3.43	5.04
			Problem 2			
1951	17.32	9.75	1.82	11.63	0.56	3.09
1955	16.92	7.76	0.90	12.25	0.0	1.50
1960	16.77	6.81	0.77	14.12	0.0	.34
			Problem 3			
1951	7.93	8.56	5.08	6.98	3.71	6.31
1955	7.36	7.40	5.26	6.72	3.10	5.75
1960	7.25	7.03	5.50	7.18	3.02	4.80
			Problem 4			
1951	7.90	8.50	3.51	6.94	2.22	4.74
1955	7.32	7.34	3.12	6.67	1.49	3.69
1960	7.20	6.96	3.42	7.13	1.36	2.96
			Problem 5			
1951	7.85	7.42	4.49	6.97	3.71	6.31
1955	7.27	6.23	3.71	6.70	3.11	5.75
1960	7.15	5.88	3.66	7.16	3.03	4.80
			Problem 6			
1951	7.83	7.39	3.56	6.93	2.27	4.80
1955	7.25	6.21	3.18	6.67	1.55	3.75
1960	7.14	5.86	3.48	7.13	1.42	3.02

Table 20. (continued)

Year	Fluid	Evaporated	Cheese	Ice Cream	Butter	Other
			Problem 7			
1951	6.04	4.04	2.45	4.26	2.86	2.16
1955	5.16	3.13	2.02	3.71	2.58	1.73
1960	5.00	2.89	1.98	3.84	2.45	1.51
			Problem 8			
1951	5.79	4.09	2.85	4.21	3.35	2.47
1955	5.09	3.16	2.13	3.68	2.71	1.82
1960	4.91	2.94	2.13	3.78	2.64	1.65
			Problem 9			
1951	6.73	4.46	2.57	4.76	2.90	2.36
1955	5.74	3.46	2.14	4.18	2.62	1.88
1960	5.55	3.18	2.10	4.35	2.48	1.63
			Problem 10			
1951	6.60	4.49	2.77	4.74	3.16	2.52
1955	5.78	3.44	2.08	4.19	2.54	1.82
1960	5.56	3.18	2.10	4.35	2.48	1.62
			Problem 11			
1951	7.42	4.89	2.69	5.26	2.95	2.55
1955	6.33	3.79	2.26	4.64	2.66	2.02
1960	6.11	3.48	2.22	4.85	2.52	1.75
			Problem 12			
1951	7.41	4.89	2.70	5.26	2.96	2.56
1955	6.48	3.73	2.03	4.70	2.37	1.83
1960	6.21	3.42	2.07	4.92	2.31	1.59
			Problem 13			
1951	8.10	5.31	2.82	5.76	2.99	2.75
1955	6.91	4.11	2.37	5.11	2.70	2.17
1960	6.67	3.77	2.34	5.36	2.55	1.87

Table 20. (continued)

Year	Fluid	Evaporated	Cheese	Ice Cream	Butter	Other
			Problem 14			
1951	8.22	5.29	2.63	5.78	2.76	2.60
1955	7.17	4.01	1.99	5.21	2.21	1.83
1960	6.86	3.66	2.03	5.49	2.15	1.57
			Problem 15			
1951	6.45	4.29	2.52	4.56	2.89	2.28
1955	5.51	3.33	2.09	3.99	2.61	1.82
1960	5.33	3.06	2.06	4.15	2.47	1.58
			Problem 16			
1951	25.30	15.94	5.87	18.28	4.09	7.67
1955	21.56	12.30	5.26	16.77	3.64	5.90
1960	20.61	11.11	5.26	17.98	3.38	4.85

Table 21. Retail prices in dollars per hundredweight for 1951, 55, and 60: actual and solutions

Year	Fluid	Evaporated	Cheese	Ice Cream	Butter
			Actual		
1951	10.69	7.94	5.39	12.50	3.94
1955	10.41	7.32	4.88	11.59	3.45
1960	10.69	7.58	4.70	10.62	3.36
			Problem 1		
1951	27.19	17.16	6.59	22.48	4.09
1955	28.46	16.64	7.09	24.09	4.15
1960	29.20	16.47	7.28	25.39	3.93
			Problem 2		
1951	23.60	13.67	3.19	19.40	1.00
1955	23.64	11.96	2.53	19.97	0.67
1960	23.90	11.37	2.33	20.93	0.64
			Problem 3		
1951	13.17	12.40	6.64	14.94	4.02
1955	13.03	11.58	7.16	14.66	3.65
1960	13.34	11.60	7.34	14.26	3.54
			Problem 4		
1951	13.14	12.33	4.98	14.90	2.59
1955	12.99	11.51	4.89	14.61	2.10
1960	13.28	11.53	5.14	14.21	1.95
			Problem 5		
1951	13.08	11.18	6.02	14.93	4.02
1955	12.93	10.33	5.51	14.64	3.66
1960	13.23	10.37	5.39	14.24	3.55
			Problem 6		
1951	13.06	11.15	5.03	14.89	2.64
1955	12.91	10.30	4.95	14.61	2.16
1960	13.22	10.35	5.20	14.21	2.00

Table 21. (continued)

Year	Fluid	Evaporated	Cheese	Ice Cream	Butter
			Problem 7		
1951	11.07	7.56	3.86	12.33	3.21
1955	10.59	7.01	3.72	11.77	3.15
1960	10.84	7.17	3.61	11.06	2.99
			Problem 8		
1951	10.80	7.62	4.28	12.28	3.68
1955	10.51	7.04	3.84	11.74	3.27
1960	10.74	7.23	3.77	11.00	3.17
			Problem 9		
1951	11.84	8.01	3.98	12.81	3.24
1955	11.23	7.36	3.35	12.22	3.19
1960	11.45	7.48	3.74	11.55	3.02
			Problem 10		
1951	11.70	8.04	4.20	12.79	3.49
1955	11.28	7.34	3.78	12.23	3.11
1960	11.46	7.48	3.74	11.55	3.02
			Problem 11		
1951	12.61	8.47	4.11	13.29	3.29
1955	11.89	7.72	3.98	12.66	3.22
1960	12.07	7.80	3.86	12.03	3.06
			Problem 12		
1951	12.60	8.47	4.12	13.29	3.30
1955	12.05	7.65	3.73	12.72	2.95
1960	12.18	7.74	3.70	12.09	2.86
			Problem 13		
1951	13.36	8.92	4.25	13.77	3.33
1955	12.53	8.06	4.09	13.12	3.26
1960	12.69	8.11	3.99	12.52	3.09

Table 21. (continued)

Year	Fluid	Evaporated	Cheese	Ice Cream	Butter
			Problem 14		
1951	13.49	8.90	4.05	13.79	3.11
1955	12.82	7.95	3.69	13.21	2.79
1960	12.90	8.00	3.66	12.64	2.70
			Problem 15		
1951	11.53	7.83	3.93	12.62	3.23
1955	10.98	7.22	3.80	12.04	3.18
1960	11.21	7.35	3.69	11.35	3.01
			Problem 16		
1951	32.45	20.30	7.48	25.79	4.39
1955	28.79	16.82	7.16	24.31	4.16
1960	28.17	15.97	7.09	24.63	3.88

Table 22. Farm quantities in million hundredweight for 1951, 1955 and 1960: actual and solutions

Year	Fluid	Evap.[a]	Cheese	I.C.[b]	Butter	Other	X-E[c]	X-C[d]	X-B[e]	Total
					Actual					
1951	541.00	64.01	116.70	70.01	303.63	51.32	4.90	-0.60	.84	1151.81
1955	575.00	58.27	141.10	81.71	324.52	48.51	4.26	-4.40	1.48	1230.45
1960	585.00	53.20	152.80	94.53	301.73	40.86	3.85	-5.00	1.06	1228.03
					Problem 1					
1951	313.36	58.56	99.18	39.70	261.73	32.04	-16.17	-7.15	0.07	781.32
1955	325.60	54.29	110.62	44.48	249.86	29.54	-16.93	-15.55	0.14	782.05
1960	328.67	50.70	117.54	50.77	236.21	25.54	-16.38	-17.85	-0.13	775.07
					Problem 2					
1951	360.60	62.89	144.49	48.32	483.01	48.37	-8.35	8.90	3.94	1152.17
1955	388.96	60.13	171.35	56.07	498.44	51.45	-6.44	5.95	4.48	1230.39
1960	398.43	57.07	183.54	63.34	471.58	49.39	-4.93	5.50	3.99	1227.91
					Problem 3					
1951	510.17	55.75	97.56	61.33	264.89	32.04	-5.49	-7.40	0.16	1009.01
1955	542.23	50.75	108.70	71.57	283.85	29.89	-5.58	-15.85	0.76	1066.32
1960	551.61	46.27	115.58	82.82	262.49	26.76	-5.46	-18.15	0.37	1062.29
					Problem 4					
1951	510.11	55.76	119.86	61.32	367.50	40.00	-5.35	0.45	1.95	1151.60
1955	542.22	50.73	139.12	71.57	394.68	40.34	-5.44	-5.15	2.69	1230.76
1960	551.76	46.27	145.14	82.82	376.77	36.09	-5.29	-7.75	2.36	1228.17
					Problem 5					
1951	510.17	58.49	105.97	61.32	264.80	32.04	-2.76	-4.45	0.16	1025.74
1955	542.19	53.53	130.81	71.57	282.96	29.89	-2.77	-8.10	0.75	1100.83
1960	551.70	48.98	141.82	82.80	261.57	26.76	-2.70	-8.95	0.35	1102.33

[a] Evaporated

[b] Ice Cream

[c] Net export of Evaporated

[d] Net export of Cheese

[e] Net export of Butter

Table 22. (continued)

Year	Fluid	Evap.[a]	Cheese	I.C.[b]	Butter	Other	X-E[c]	X-C[d]	X-B[e]	Total
					Problem 6					
1951	510.10	58.46	119.14	61.34	364.04	39.70	-2.69	0.20	1.89	1152.18
1955	542.19	53.48	138.26	71.57	390.52	40.04	-2.72	-5.45	2.62	1230.51
1960	551.60	48.95	144.27	82.81	372.62	35.79	-2.65	-8.05	2.29	1227.63
					Problem 7					
1951	534.97	64.83	134.83	68.75	322.86	53.09	5.35	5.75	1.18	1191.61
1955	572.16	58.86	154.68	79.80	318.95	50.28	4.67	0.35	1.38	1241.13
1960	582.40	53.99	165.53	91.96	300.99	43.45	4.47	-0.55	1.05	1243.29
					Problem 8					
1951	539.15	64.46	129.14	68.92	289.07	51.52	5.23	3.75	0.59	1151.83
1955	573.34	58.71	153.11	79.89	309.98	49.83	4.60	-0.20	1.23	1230.49
1960	583.93	53.77	163.40	92.14	287.88	42.74	4.35	-1.30	0.82	1227.73
					Problem 9					
1951	524.43	64.55	133.17	67.38	320.21	52.08	4.35	5.15	1.13	1172.45
1955	563.28	58.68	153.02	78.51	316.28	49.52	3.88	-0.25	1.34	1224.26
1960	573.96	53.87	163.87	90.55	299.01	42.84	3.78	-1.15	1.01	1227.74
					Problem 10					
1951	526.60	64.34	130.33	67.45	302.28	51.27	4.27	4.15	0.82	1151.51
1955	562.60	58.77	153.87	78.47	321.80	49.83	3.92	0.05	1.43	1230.74
1960	573.80	53.89	163.87	90.55	299.01	42.89	3.78	-1.15	1.01	1227.65
					Problem 11					
1951	513.90	64.24	131.52	66.01	316.87	51.11	3.31	4.55	1.07	1152.58
1955	554.25	58.51	151.35	77.24	313.62	48.81	3.08	-0.85	1.29	1207.30
1960	565.38	53.75	162.20	89.17	296.35	42.23	3.06	-1.75	0.97	1211.36
					Problem 12					
1951	514.06	64.23	131.37	66.01	316.18	51.06	3.31	4.50	1.06	1151.78
1955	551.72	58.81	154.62	77.06	333.62	49.78	3.23	0.30	1.64	1230.78
1960	563.67	54.00	164.33	88.97	310.83	43.04	3.20	-1.00	1.22	1228.26

Table 22. (continued)

Year	Fluid	Evap.[a]	Cheese	I.C.[b]	Butter	Other	X-E[c]	X-C[d]	X-B[e]	Total
Problem 13										
1951	503.52	63.95	129.72	64.63	314.22	50.10	2.31	3.90	1.02	1133.37
1955	545.36	58.36	149.83	75.95	310.96	48.05	2.32	-1.40	1.24	1190.67
1960	556.78	53.65	160.54	87.77	294.38	41.62	2.36	-2.35	0.93	1195.68
Problem 14										
1951	501.52	64.11	132.42	64.56	330.08	50.86	2.35	4.85	1.30	1152.05
1955	540.98	58.86	155.23	75.64	344.75	49.78	2.56	0.50	1.83	1230.13
1960	553.54	54.11	164.94	87.38	321.96	44.41	2.63	-0.80	1.41	1228.32
Problem 15										
1951	528.71	64.66	133.87	67.93	320.86	52.48	4.75	5.40	1.14	1179.80
1955	566.80	58.75	153.71	79.03	316.93	49.83	4.19	0.00	1.35	1230.59
1960	577.33	53.93	164.42	91.11	299.66	43.09	4.07	-0.95	1.03	1233.69
Problem 16										
1951	241.02	56.50	87.58	30.22	241.11	25.14	-23.21	-11.35	-0.30	646.71
1955	321.01	54.20	109.79	43.85	248.53	29.13	-17.34	-15.85	0.11	773.43
1960	342.96	50.84	120.04	52.94	239.50	26.51	-15.25	-16.95	-0.07	800.52

Table 23. Cash receipts in millions of dollars for 1951, 1955, and 1960: actual and solutions

Year	Fluid	Evap.[a]	Cheese	I.C.[b]	Butter	Other	X-E[c]	X-C[d]	X-B[e]	Total
					Actual					
1951	3078.29	270.76	434.12	268.14	953.40	128.81	20.73	-2.23	2.64	5154.65
1955	2875.00	192.29	419.07	248.40	811.30	100.90	14.06	-13.07	3.70	4651.64
1960	2843.10	167.58	438.54	276.97	736.22	82.54	12.13	-14.35	2.59	4545.30
					Problem 1					
1951	6442.68	761.87	498.88	588.75	989.34	202.17	-210.37	-35.96	0.26	9237.59
1955	6922.25	658.54	575.22	735.70	904.49	171.92	-205.36	-80.86	0.51	9682.39
1960	7079.55	587.11	639.42	952.95	810.20	128.72	-189.68	-97.10	-0.45	9910.70
					Problem 2					
1951	6245.59	613.18	262.96	561.96	270.49	149.46	-81.41	16.20	2.21	8040.62
1955	6581.20	466.61	154.21	686.86	0.0	77.17	-49.97	5.35	0.0	7921.42
1960	6681.66	388.65	141.33	894.36	0.0	16.79	-33.57	4.23	0.0	8093.44
					Problem 3					
1951	4045.65	477.22	495.60	428.08	982.74	202.17	-46.99	-37.59	0.59	6547.46
1955	3990.81	375.55	571.76	480.95	879.93	171.87	-41.29	-83.37	2.36	6348.56
1960	3999.17	325.28	635.69	594.65	792.72	128.45	-38.38	-99.82	1.12	6338.85
					Problem 4					
1951	4029.87	473.96	420.71	425.56	815.85	189.60	-45.47	1.58	4.33	6315.96
1955	3969.05	372.36	434.05	477.37	588.07	148.85	-39.33	-16.07	4.01	5937.76
1960	3972.67	322.04	496.38	590.51	512.41	106.83	-36.82	-26.50	3.21	5940.70
					Problem 5					
1951	4004.83	434.00	475.81	427.40	982.41	202.17	-20.48	-19.98	0.59	6486.74
1955	3941.72	333.49	485.30	479.52	880.01	171.87	-17.26	-30.05	2.33	6246.93
1960	3944.65	288.00	519.06	592.85	792.56	128.45	-15.88	-32.76	1.06	6217.98

[a] Evaporated

[b] Ice Cream

[c] Net export of Evaporated

[d] Net export of Cheese

[e] Net export of Butter

Table 23. (continued)

Year	Fluid	Evap.[a]	Cheese	I.C.[b]	Butter	Other	X-E[c]	X-C[d]	X-B[e]	Total
					Problem 6					
1951	3994.08	432.02	424.14	425.09	826.37	190.56	-19.88	0.71	4.29	6277.37
1955	3930.88	332.11	439.67	477.37	605.31	150.15	-16.89	-17.33	4.06	5905.30
1960	3938.42	286.85	502.06	590.44	529.12	108.09	-15.53	-28.01	3.25	5914.66
					Problem 7					
1951	3231.22	261.91	330.33	292.87	923.38	114.67	21.61	14.09	3.37	5193.46
1955	2952.34	184.23	312.45	296.06	822.89	86.98	14.62	0.71	3.56	4673.83
1960	2912.00	156.03	327.75	353.13	737.43	65.61	12.92	-1.09	2.57	4566.34
					Problem 8					
1951	3121.68	263.64	368.05	290.15	968.38	127.25	21.39	10.69	1.98	5173.20
1955	2918.30	185.52	326.12	294.00	840.05	90.69	14.54	-0.43	3.33	4672.11
1960	2867.10	158.08	348.04	348.29	760.00	70.52	12.79	-2.77	2.16	4564.21
					Problem 9					
1951	3529.41	287.89	342.25	320.73	928.61	122.91	19.40	13.24	3.28	5567.70
1955	3233.23	203.03	327.46	328.17	828.65	93.10	13.42	-0.53	3.51	5030.03
1960	3185.48	171.31	344.13	393.89	741.54	69.83	12.02	-2.41	2.50	4918.28
					Problem 10					
1951	3475.56	288.89	361.01	319.71	955.20	129.20	19.17	11.50	2.59	5562.82
1955	3251.83	202.17	320.05	328.79	817.37	90.69	13.48	0.10	3.63	5028.11
1960	3190.33	171.37	344.13	393.89	741.54	69.43	12.02	-2.41	2.50	4922.84
					Problem 11					
1951	3813.14	314.13	353.79	347.21	934.77	130.33	16.19	12.24	3.16	5924.94
1955	3508.40	221.75	342.05	358.39	834.23	98.60	11.67	-1.92	3.43	5376.60
1960	3454.47	187.05	360.08	432.47	746.80	73.90	10.65	-3.88	2.44	5263.98
					Problem 12					
1951	3809.18	314.08	354.70	347.21	935.89	130.71	16.19	12.15	3.14	5923.24
1955	3575.15	219.36	313.88	362.18	790.68	91.10	12.05	0.61	3.89	5368.87
1960	3500.39	184.68	340.16	437.73	718.02	68.43	10.94	-2.07	2.82	5261.09

Table 23. (continued)

Year	Fluid	Evap.[a]	Cheese	I.C.[b]	Butter	Other	X-E[c]	X-C[d]	X-B[e]	Total
					Problem 13					
1951	4078.51	339.57	365.81	372.27	939.52	137.77	12.27	11.00	3.05	6259.75
1955	3768.44	239.86	355.10	388.10	839.59	104.27	9.54	-3.32	3.35	5704.91
1960	3713.72	202.26	375.66	470.45	750.67	77.83	8.90	-5.50	2.37	5596.35
					Problem 14					
1951	4122.49	339.14	348.26	373.16	911.02	132.24	12.43	12.76	3.59	6255.07
1955	3878.83	236.03	308.91	394.08	761.90	91.10	10.27	0.99	4.04	5686.12
1960	3797.28	198.04	334.83	479.72	692.21	67.75	9.63	-1.62	3.03	5580.85
					Problem 15					
1951	3410.18	277.39	337.35	309.76	927.28	119.65	20.38	13.61	3.29	5418.89
1955	3123.07	195.64	321.25	315.33	827.19	90.69	13.95	0.0	3.52	4890.63
1960	3077.17	165.03	338.71	378.11	740.16	68.08	12.45	-1.96	2.54	4780.29
					Problem 16					
1951	6097.80	900.61	514.09	552.42	986.14	192.82	-369.97	-66.62	-1.23	8806.05
1955	6920.97	666.66	577.50	735.36	904.65	171.87	-213.28	-83.37	0.40	9680.73
1960	7068.40	564.83	631.41	951.86	809.51	128.57	-169.43	-89.16	-0.24	9895.75